Multimedia Video Coding & Architectures (5LSE0), Module 02

Measure of Information

Coding of Discrete Sources

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Multimedia Video Coding and Architectures (5DD40), Module 02

Mod 02, Part 1

Probability and Information

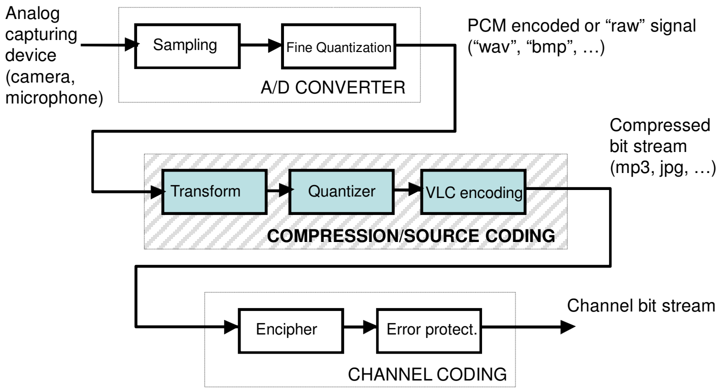
Questions to be Answered

Three questions play a central role:

1. Why can signals be compressed?
2. How much can signals be compressed?
3. Which signal processing / information theory algorithms are most efficient in reaching the maximum compression?

(For lossless and *lossy* compression)

System Overview / Embedding compression



Why can Signals be Compressed? – (1)

Because signal amplitudes are mutually dependent

Question 1:

What is the *best possible* exploitation of the correlation (dependencies) in natural signals?

(Rate-Distortion Theory)

Question 2:

How do we *implement* a system that exploits the correlation in natural signals?

(Compression algorithms: - DPCM
- Subband/wavelet
- Transform/DCT
- Motion compensation)

Why can Signals be Compressed? – (2)

Because infinite accuracy of signal amplitudes is (perceptually) irrelevant

Question 1:

What is the *best possible* trade-off between required bit rate and resulting distortion?

(Rate-Distortion Theory)

Question 2:

How do we *implement* a system that gives us that best possible trade-off?

(Scalar and Vector Quantization Theory)

Why can Signals be Compressed? – (3)

Because signal amplitudes are statistically redundant

Question 1:

What is the *shortest* average codeword length that one can achieve for a given signal (or "source")?

(Shannon Information Theory)

Question 2:

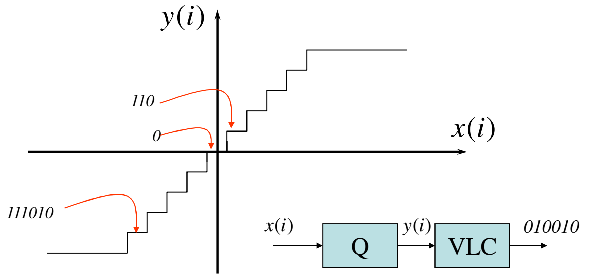
How did you obtain those codewords?

(Construction Recipes)

Relevance of Coding Subject /

all corner stones can be re-used…

- Theory is more generally applicable
- As a start, keep in mind that we are discussing how to convert the output of a quantizer to codewords



Quantifying Information & Coding

- **What does "statistical redundancy" mean?**
  - (example illustrated: some amplitudes/symbols/ events are *more probable* than others)
  - Need to quantify this concept
- **Foundation laid in 1948 by Claude Shannon**

  ("A Mathematical Theory of Communication", Bell Syst. Techn. Journal)
- **Based on formal definition of concept "Information" or "Entropy"**

What is "Information" / Definition …

- **Interpretation of "Information"**
  - Philosophy
  - Psychology
  - Biology
  - Engineering

  semantic
  syntactic
  pragmatic
- **Think of examples in "natural languages"**
  - Often has to do with the surprise or uncertainty a message contains
  - Some messages give a lot of information, others no information at all
- **Interested in** the mathematical definition

Example of information: Lottery

- **Case 1**: Two people put €10 each on the table. A fair coin is flipped, the winner takes all
- **Case 2A**: 1024 people put €10 each on the table. A number between 1 and 1024 is drawn randomly, the winner takes all.
- **Case 2B**: 1024 people put €10 each on the table, a fair coin is flipped
  - If head, I take all money
  - If tail, I loose. Then a number between 1 and 1023 is drawn randomly, the winner takes all
- **Which of these cases is most surprising ~ contains most information?**

Information and Probability

- **The *less likely* an event or *symbol* s_i is, the more *uncertainty* exists, and the *more information* one obtains if this event/symbol occurs**

$$I(s_i) = -\log_2[P_S(s_i)] \qquad \text{(bit of information)}$$

Case 1:	*I*(win)	= 1 bit
Case 2A:	*I*(win)	= 10 bits
Case 2B:	*I*(I win)	= 1 bit
	I(you win)	≈ 11 bits

Another Example / 8-message source – (1)

- Amount of information if one observes a signal amplitude

Probability of occurrence

s_i	$P_S(s_i)$	$I(s_i)$
0	*0.125*	*3 bits*
1	*0*	
2	*0.5*	*1 bit*
3	*0*	
4	*0.125*	*3 bits*
5	*0.125*	*3 bits*
6	*0*	
7	*0.125*	*3 bits*

Shannon's Measure of Information

- **We are interested in the *average* amount of information that one observes per symbol (*average amount of information that a quantizer produces*)**

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bit/symbol)}$$

- **Commonly known as**
  - "p log p" information measure
  - ***Entropy*** of a source (quantizer) ~ *chaos in physics*

Lottery Example / Sense of information – (2)

- **The *more equally* probable the events or symbols s_i are, the *more uncertainty* exist, and the *more information* is obtained**

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bit)}$$

- **Case 1:** $H(S)$ = 1 bit
- **Case 2A:** $H(S)$ = 10 bits
- **Case 2B:** $H(S)$ $\approx$ 6 bits

Another Example / 8-message source – (2)

s_i	$P_S(s_i)$	$I(s_i)$
0	*0.125*	*3 bits*
1	*0*	
2	*0.5*	*1 bits*
3	*0*	
4	*0.125*	*3 bits*
5	*0.125*	*3 bits*
6	*0*	
7	*0.125*	*3 bits*

$H(S)$ = 2 bits of information per amplitude

Properties of entropy *H*(*S*)

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bits/symb.)}$$

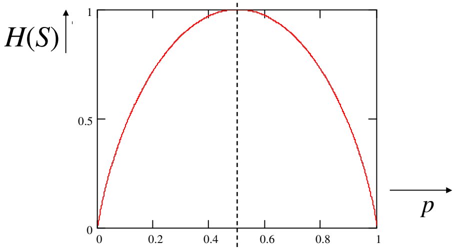
The definition of Information holds for zero-memory (memoryless) discrete sources

- ***H*(*S*) is positive**
- ***H*(*S*) is continuous in symbol probabilities**
- ***H*(*S*) is symmetric**
- ***H*(*S*) is maximum, if all symbol probabilities are equal**

$$0 \le H(S) \le \log_2(N)$$

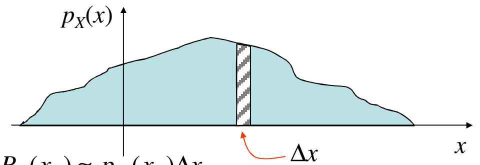
Entropy example for a Binary Source

- **Two symbols s_0 and s_1**
- $P[s_0] = p \qquad P[s_1] = 1\text{-}p$
- $H(S) = -p\log_2 p - (1\text{-}p)\log_2(1\text{-}p)$



What about Continuous Sources? – (1)

- **Generalize Shannon's information measure for the case that source symbols (signal amplitudes) are continuous**



$$P_X(x_i) \approx p_X(x_i)\Delta x$$

$$H(X) = \lim_{\Delta x\to 0}\left\{-\sum_i p_X(x_i)\Delta x\log_2[p_X(x_i)\Delta x]\right\}$$

What about Continuous Sources? – (2)

- **Result:**

$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx - \underbrace{\lim_{\Delta x\to 0}\log_2\Delta x}_{\textit{goes to infinite (as expected?)}}$$

- **Therefore we use differential entropy:**

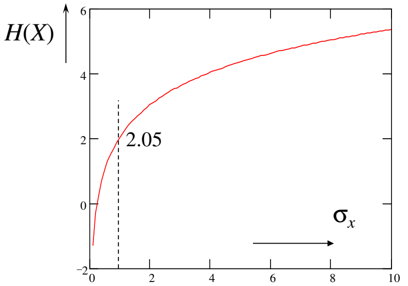
$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx$$

- **Note: $H(X)$ may become negative! Interpretation of entropy is "less obvious" for continuous sources**

Differential Entropy Gaussian Signal

- For memoryless Gaussian sources (i.e. $p_X(x)$ is Gaussian)

$$H(X) = \frac{1}{2}\log_2(2\pi e) + \log_2(\sigma_x)$$



Theory and practice… memoryless sources

- The expressions

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bit)}$$

$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx$$

hold for memoryless sources.

- The audio/image/video data we wish to compress are usually not memoryless!
- However, we will see that the “Transform block” will try to do just that

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Back to Discrete Sources

- Important application of Shannon’s entropy measure is in finding efficient (~ short average length) code words
- The entropy $H(S)$ tells us what the ***minimal average code word length*** is of any
  - instantaneously decodable
  - uniquely decodable
  - nonsingular
  - binary block

code that can be designed for the source ***S***

(without telling how to find that code)

Multimedia Video Coding and Architectures (5DD40), Module 02

Mod 02, Part 2

Coding: Definitions & Examples

Coding / Technical terms Explained

- ***Binary block code***
  - Each symbol s_i is mapped into a *fixed* binary code word
- ***Nonsingular code***
  - Different symbols map into different code words
- ***Uniquely decodable***
  - From the concatenation of code words, the symbols can uniquely be recovered. Example
- ***Instantaneously decodable***
  - Code words can be decoded without reference to the next code word. Example

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Example Uniquely Decodable

Not uniquely decodable

s_1 : 0

s_2 : 11

s_3 : 00

s_4 : 01

$s_1 s_3$: 000

$s_3 s_1$: 000

Some concatenations lead to a singular code!

Uniquely decodable

s_1 : 0

s_2 : 10

s_3 : 110

s_4 : 111

Any combination of symbols can be uniquely decoded (any concatenation leads to a nonsingular code)

Example Instantaneously Decodable

Non-instantaneous

s_1 : 0

s_2 : 01

s_3 : 011

s_4 : 0111

$s_1 s_3 s_4$: 00110111

Need to observe next "0" to know that previous bit ended a code word

Instantaneous

s_1 : 0

s_2 : 10

s_3 : 110

s_4 : 1110

$s_1 s_3 s_4$: 01101110

By observing this "0" we immediately know a code word was found

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Noiseless Source Coding Theorem

- **For a zero-memory discrete source with entropy**

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bit)}$$

an (instantaneously and uniquely decodable, nonsingular block) binary code exists for which the average code word length L is

$$H(S) \le L \le H(S) + 1$$

Entropy has even better bound for L …

- If we take M independent symbols together, then

$$H(S) \leq L \leq H(S) + \frac{1}{M}$$

- In other words, at sufficient expense ($M \to \infty$), a code exists of which the average code word length is arbitrarily close to the entropy of the source.

Example – 8-message source

s_i	$P_S(s_i)$
0	0.005
1	0.02
2	0.14
3	0.20
4	0.51
5	0.08
6	0.04
7	0.005

- Simple binary coding requires 3 bits/symbol
- *H*(*S*) = 2.024 bits/symbol (creates clear reduction!)

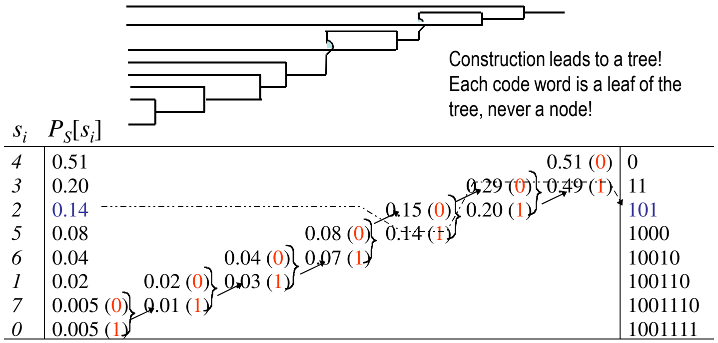
How to find that Code…?

- A number of construction recipes are know, usually named after their "inventor"
  - Shannon-Fano code
  - Gilbert-Moore code
  - Arithmetic code
  - Huffman code
    - This one is used very often in compression
    - Often in combination with run-length code

Huffman Binary Code Construction – (1)

1. Rank the symbols with decreasing probability
2. Join the two least probable symbols and add their probabilities to form a new "joined symbol"
3. Re-arrange new set of probabilities in decreasing order
4. Repeat Step 2 and 3 until two probabilities remain
5. Assign a bit "0" to one of the probabilities, and a bit "1" to the other
6. Go backwards and add one bit at each place where two symbols were joined
7. Create code words following the path to that symbol from right to left

Huffman Binary Code Construction – (2)



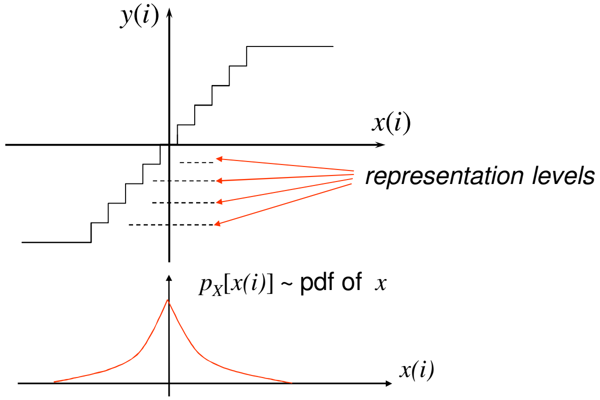
Construction leads to a tree!
Each code word is a leaf of the tree, never a node!

Huffman Binary Code Construction – (3)

s_i	$P_S(s_i)$	Codeword
0	*0.005*	*1001111*
1	*0.02*	*100110*
2	*0.14*	*101*
3	*0.20*	*11*
4	*0.51*	*0*
5	*0.08*	*1000*
6	*0.04*	*10010*
7	*0.005*	*1001110*

- **Simple binary coding requires 3 bits/symbol**
- $H(S)$ **= 2.024 bits/symbol**
- L_{av} **= 2.204 bits/symbol (we approach closely the bound: … entropy)**

Quantization / Example – (1)



Quantization / Example – (2)

	Repres. Level	Probability	Codewords (positive)	Codewords (negative)
1	±9.247	0.00007	00010101101010	00010101101011
2	±6.875	0.00031	000101011011	0001010110100
3	±5.519	0.00077	0001010111	00010101100
4	±4/562	0.00152	0001010000	0001010001
5	±3.823	0.00269	000101001	000101010
6	±3.217	0.00444	00011010	00011011
7	±2.703	0.00699	1011010	1011011
8	±2.253	0.01068	0001011	0001100
9	±1.855	0.01590	000111	101100
10	±1.497	0.02318	10111	000100
11	±1.175	0.03316	01000	01001
12	±0.884	0.04658	1100	1101
13	±0.622	0.06441	0101	1010
14	±0.388	0.08797	111	0000
15	±0.180	0.11987	011	100
16	0.000	0.16292	001	

- **Simple binary coding requires 5 bits/repr.level**
- $H(S)$ **= 3.876 bits/repr.level**
- L_{av} **= 3.912 bits/repr.level**

Limitations Huffman Code / Modifications

- **Length L always larger than 1.0 bit/symbol**
- **Predesign of the code is necessary**
- **Fixed code table**
- **If probabilities are different than the ones used in the design, data *expansion* may occur**
- **Alternative versions:**
  - Two-pass implementation
  - Block adaptive (code table per block of data)
  - Recursive Huffman (changes code table continuously)

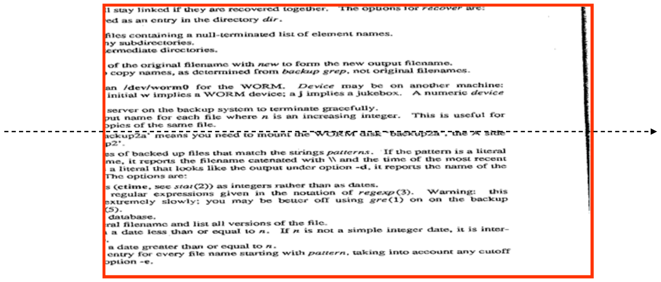
What if statistical dependencies exist?

- **So far, source symbols were assumed independent**
  - We considered only discrete *memoryless* sources
  - Quantizer representation levels were considered one-by-one.

- **What if the source symbols are dependent?**
  - Model that dependency and design codes (~document compression, Markov chains, "context coders")
  - Solution: use ***runlength*** coding

Runlength Coding – (1)

- **Useful for coding of "long runs of the same symbol"**
- **Example: Binary picture (fax)**



l stay linked if they are recovered together. The options for *recover* are:
ed as an entry in the directory *dir*.

files containing a null-terminated list of element names.
ny subdirectories.
ermediate directories.

of the original filename with *new* to form the new output filename.
copy names, as determined from *backup grep*, not original filenames.

an /dev/worm0 for the WORM. *Device* may be on another machine:
initial w implies a WORM device; a j implies a jukebox. A numeric *device*

server on the backup system to terminate gracefully.
put name for each file where *n* is an increasing integer. This is useful for
opies of the same file.
ackup2a' means you need to mount the WORM disk "backup2a", the A side
p2'.

es of backed up files that match the strings *patterns*. If the pattern is a literal
me, it reports the filename catenated with \\ and the time of the most recent
a literal that looks like the output under option -d, it reports the name of the
The options are:

s (ctime, see *stat*(2)) as integers rather than as dates.
regular expressions given in the notation of *regexp*(3). Warning: this
extremely slowly; you may be better off using *gre*(1) on on the backup
(5).
database.
ral filename and list all versions of the file.
a date less than or equal to *n*. If *n* is not a simple integer date, it is inter-

a date greater than or equal to *n*.
entry for every file name starting with *pattern*, taking into account any cutoff
option -e.

Runlength Coding – (2)

- **Representing**
  - Black = "1", White = "0"

we get for instance

"0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 1 1 1 0 0 ..."

- **Now code the "runs" of identical symbols**
  - **8** ("zeroes") **2** ("ones") **6** ("zeroes") **5** ("ones") …
- **The runlengths themselves are encoded by bit patterns**
  - For instance (separated?) Huffman code

Application to Signal Coding

- Later, we see that efficient "transforms" used in compression produce
  - A lot of "zero" values
  - Some (significant) non-zero values
- Typical symbol sequences to be coded:
  "5 1 0 0 0 0 0 0 0 3 0 0 6 0 0 0 0 1 0 0 0 0 …"
  - Will be done by {zero-run, non-zero symbol} pairs
  - Here: "{0,5}, {0,1}, {7,3}, {2,6}, {4,1}, …"
  - The pairs are assigned a Huffman code

JPEG Coding / Example

"run of zeros"

"non-zero symbol"
-> amplitude class

Zero run length	Amplitude Category	Code length	Codeword
0	1	2	00
0	2	2	01
0	3	3	100
0	4	4	1011
0	5	5	11010
0	6	6	111000
0	7	7	1111000
1	1	4	1100
1	2	6	111001
1	3	7	1111001
1	4	9	111110110
2	1	5	11011
2	2	8	11111000
3	1	6	111010
3	2	9	111110111
4	1	6	111011
5	1	7	1111010
6	1	7	1111011
7	1	8	11111001
8	1	8	11111010
9	1	9	111111000
10	1	9	111111001
End of Block (EOB)		4	1010

Why can Signals be Compressed?

Because ~~signal amplitudes are~~ statistically redundant
quantizer representation levels

Question 1:
What is the *shortest* average codeword length that one can achieve for a given signal (or "source")?

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \quad \text{(bit)}$$

$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx$$

Question 2:
How did you obtain those codewords?

Huffman Coding
Run-length coding

The following slides were used earlier and are included for reference only

Back to Quantification

Why can Signals be Compressed?

Because signal amplitudes are statistically redundant

Entropy Coding or Variable Length Coding

Example**:**

- Signal has amplitudes in range [0,7]
- **2** bits per signal sample

Signal value	Probability	Codeword
0	*0.125*	*100*
1	*0*	
2	*0.5*	*0*
3	*0*	
4	*0.125*	*101*
5	*0.125*	*110*
6	*0*	
7	*0.125*	*111*

Back to Source Coding Theorem

Why can Signals be Compressed?

Because signal amplitudes are statistically redundant

Average codeword length *L*:

Probability of symbol

$$L = \sum_{i=1}^{N} P_S(s_i) l_i$$

"#Symbols"

Length of codeword

Signal value	$P_S(s_i)$	Codeword	l_i
0	*0.125*	*100*	*3*
1	*0*		-
2	*0.5*	*0*	*1*
3	*0*		-
4	*0.125*	*101*	*3*
5	*0.125*	*110*	*3*
6	*0*		-
7	*0.125*	*111*	*3*

Back to Runlength Coding

Why can Signals be Compressed?

Because signal amplitudes are mutually dependent