Multimedia Video Coding & Architectures (5LSE0), Module 03

Scalar Quantization

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slides version 1.0

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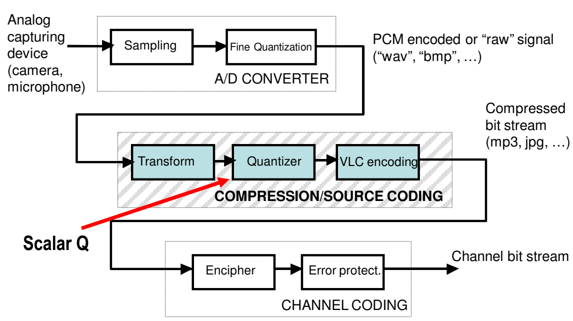
Mod 03, Preliminaries

Brief introduction: where are we…?

System Overview



Why can Signals be Compressed? – (1)

Because ~~signal amplitudes~~ are statistically redundant

(When Quantizing:) quantizer representation levels

Question 1:
What is the *shortest* average codeword length that one can achieve for a given signal (or "source")?

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \quad \text{(bit)}$$

$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx$$

Question 2:
How did you obtain those codewords?

Huffman Coding
Run-length coding

Why can Signals be Compressed? – (2)

Because signal amplitudes are mutually dependent

Question 1:
What is the *best possible* exploitation of the correlation (dependencies) in natural signals?
(Rate-Distortion Theory)

Question 2:
How do we *implement* a system that exploits the correlation in natural signals?
(Compression algorithms: - DPCM
- Subband/wavelet
- Transform/DCT
- Motion compensation)

To be discussed later in course

Why can Signals be Compressed? – (3)

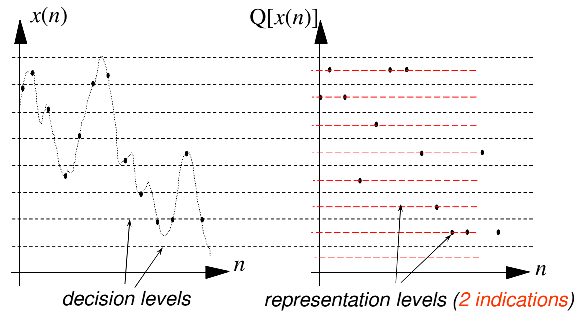
Because infinite accuracy of signal amplitudes is (perceptually) irrelevant

Question 1:
What is the *best possible* trade-off between required bit rate and resulting distortion?
(Rate-Distortion Theory)

Question 2:
How do we *implement* a system that gives us that best possible trade-off?
(Scalar and Vector Quantization Theory)

5LSE0 - Mod 03
Part 1
Characterization of Quantizers

Quantization Process – (1)



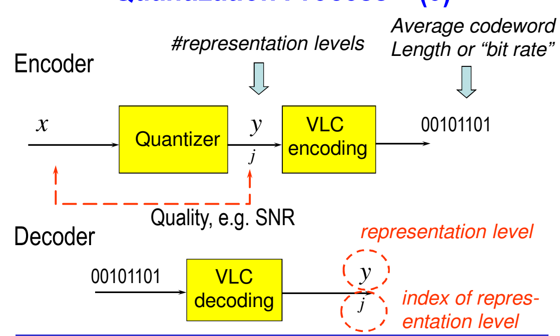
Quantization Process – (2)

- **Quantization: Map a continuous-valued signal value $x(n)$ onto a (limited set of) discrete-valued signals values $y(n)$:**

$$y(n) = Q[x(n)]$$

  such that $y(n)$ is good approximation of $x(n)$.
- **Important: # bits to represent $y(n)$.**
  (= average codeword length per representation level)
- **Design/Optimization Problem**
  - What positions of the "*decision levels*"
  - What positions of the "*representation levels*"

Quantization Process – (3)

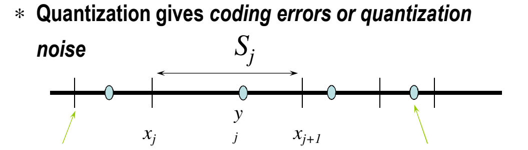


Formal Quantization Definition

- **Given amplitude x, divide domain of x into K non-overlapping intervals S_k:**

$$S_k = \{x \mid x_k < x \leq x_{k+1}\} \qquad k = 1,2,\dots,K$$

- **If x falls in S_j then it is represented by y_j.**
- **Quantization gives *coding errors or quantization noise***



Quantizer Characteristics – (1)

- **Signals values are assumed to lie symmetrically around zero**

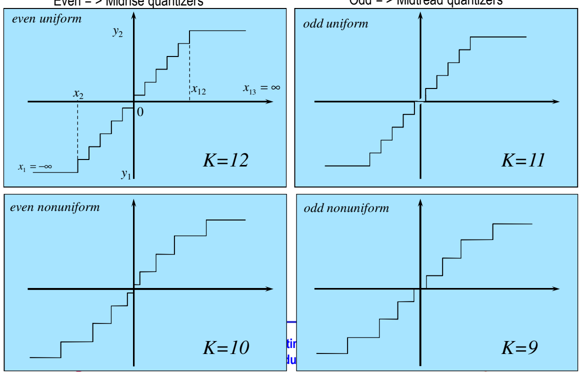
⇨ **Quantizers are usually *symmetric about origin***

- **Quantizer choices:**
  - Odd or even number of levels? } *"structure"*
  - Uniform or non-uniform quantizers?
  - Way of optimizing the quantizer? } *"design"*

Quantizer Characteristics – (2)

Even = > Midrise quantizers

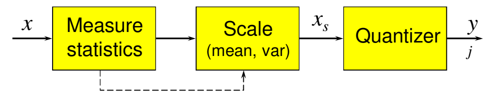
Odd = > Midtread quantizers



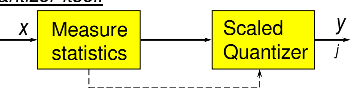
Preprocessing for Quantization – (1)

- **Only *prototype* quantizers are designed**
  - Signal is assumed to be zero-mean, unity variance

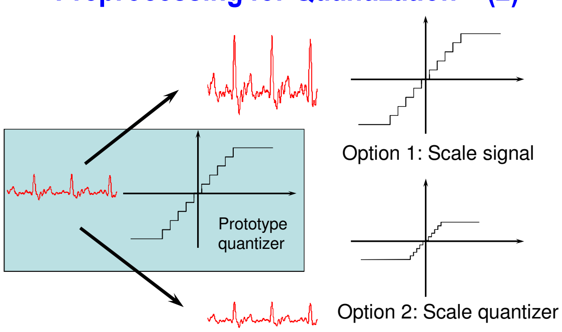
Scale signal values to be quantized



Scale quantizer itself



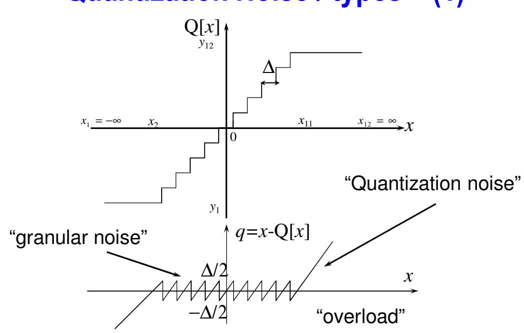
Preprocessing for Quantization – (2)



Quantizer Characteristics – (3)

- **Design of quantizer is optimization problem**
  - Find optimal values for decision and representation levels
- **Two approaches:**
  - Exploit knowledge of probability density function (PDF) of *x* to find optimum given #representation levels
    - Uniform quantization: easily solved and implemented
    - Non-uniform quantization: more difficult with limited gain
  - Select uniform quantizer with variable coarseness; determine effective #representation levels during applications
    - More practical; this is what JPEG/MPEG do
    - Choices inspired by theory

Quantization Noise / types – (1)

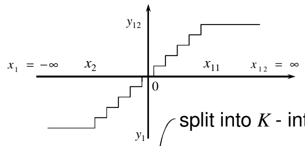


Quantization Noise / Variance – (2)

- **Quantification of (average) quantization error:**
  Variance of the quantization noise: $q = x - Q[x]$
- **Need to model probability density of** x: $p_X(x)$
- **Quantization noise variance:**

$$\sigma_q^2 = \int_{-\infty}^{\infty} \underbrace{(x - Q[x])^2}_{\text{Amount of error}} \underbrace{p_X(x)\,dx}_{\text{Probability of amount}}$$

Quantization Noise / Variance – (3)



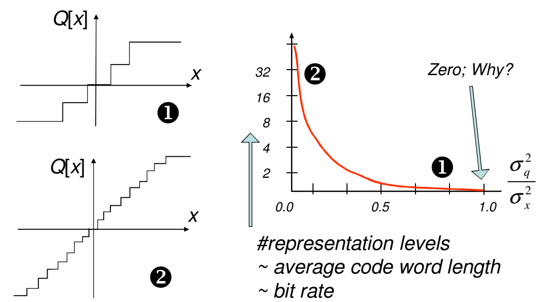
split into K - intervals

$$\sigma_q^2 = \int_{-\infty}^{\infty} (x - Q[x])^2\, p_X(x)\,\mathrm{d}x = \sum_{k=1}^{K} \int_{x_k}^{x_{k+1}} (x - y_k)^2\, p_X(x)\,\mathrm{d}x$$

Normalized measure: Signal-to-Noise-Ratio (SNR):

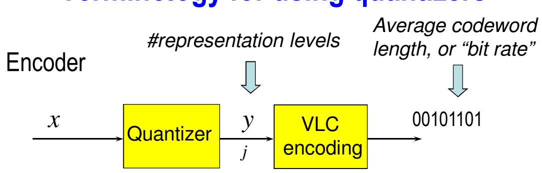
$$SNR = 10\log_{10}\frac{\sigma_x^2}{\sigma_q^2}\ \text{(dB)}$$

Bit Rate versus Distortion – (1)



#representation levels
~ average code word length
~ bit rate

Terminology for using quantizers



* **Representation levels coded in $\lceil \log_2(K) \rceil$ bits:** ***Fixed-rate quantization***
* **Representation levels coded in $H(Y)$ bits:** ***Quantization with entropy encoding***
* **(Later: *Entropy-constrained quantization*)**

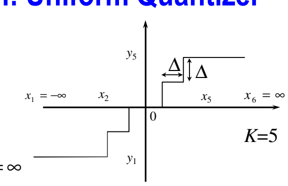
Bit Rate versus Distortion – (2)

* **Performance of quantizer is determined by**
  - the number of representation levels (bit rate or average codeword length R)
  - the quality σ_q^2 or SNR
* **Fixed-rate quantizer design:**

  For given K, find the quantizer characteristic with smallest σ_q^2

5 LSE0 - Mod 03, Part 2
Uniform Quantization

R-D Optimal Design: Uniform Quantizer – (1)



K=5

* **Given:**

  $x_1 = -\infty \qquad x_{L+1} = \infty$

  $x_{k+1} - x_k = \Delta \qquad y_k = \dfrac{(x_{k+1} + x_k)}{2}$

* **Find Δ such that σ_q^2 is minimized**

25

R-D Optimal Design: Uniform Quantizer – (2)

$$\min_{\Delta} \sum_{k=1}^{K} \int_{x_k}^{x_{k+1}} (x - y_k)^2 \, p_X(x) \, \mathrm{d}x$$

For values of K>3, this requires numerical optimization

$\sigma_x = 1$

R (bit/sample)	Δ/σ_x			SNR (dB)		
	Uniform	Gaussian	Laplace	Uniform	Gaussian	Laplace
1	1.732	1.596	1.414	6.02	4.40	3.01
2	0.866	0.996	1.087	12.04	9.25	7.07
3	0.433	0.586	0.731	18.06	14.27	11.44
4	0.217	0.335	0.461	24.08	19.38	15.96
5	0.108	0.188	0.280	30.10	24.57	20.60
6	0.054	0.104	0.166	36.12	29.83	25.36
7	0.027	0.057	0.096	42.14	35.13	30.23
8	0.013	0.031	0.055	48.17	40.34	35.14

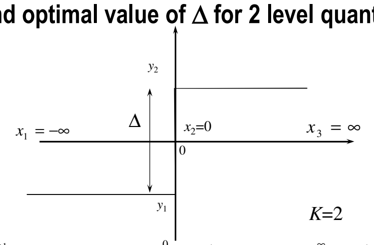
smaller D: "finer" quantizer

more difficult to quantize

26

Example: Two-Level Quantizer – (1)

* Find optimal value of Δ for 2 level quantizer



$$\sigma_q^2 = \sum_{k=1}^{K} \int_{x_k}^{x_{k+1}} (x - y_k)^2 p_X(x)\,\mathrm{d}x = \int_{-\infty}^{0} (x + \frac{\Delta}{2})^2 p_X(x)\,\mathrm{d}x + \int_{0}^{\infty} (x - \frac{\Delta}{2})^2 p_X(x)\,\mathrm{d}x$$

27

Two-Level Quantizer – (2)

$$\sigma_q^2 = \int_{-\infty}^{0} (x + \frac{\Delta}{2})^2 p_X(x)\,\mathrm{d}x + \int_{0}^{\infty} (x - \frac{\Delta}{2})^2 p_X(x)\,\mathrm{d}x$$

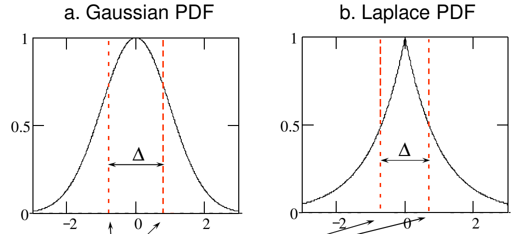
$$= \sigma_x^2 + \frac{\Delta^2}{4} - 2\Delta \int_0^{\infty} x p_X(x)\,\mathrm{d}x$$

$$\Rightarrow \Delta_{\text{optimal}} = \min_{\Delta} \sigma_q^2 = 4\int_0^{\infty} x p_X(x)\,\mathrm{d}x = 2E[|X|]$$

PDF	Δ/σ_x	σ_q^2	SNR (dB)
Uniform	1.732	0.250	6.02
Gaussian	1.596	0.363	4.40
Laplace	1.414	0.500	3.01
Gamma	1.154	0.667	1.76

28

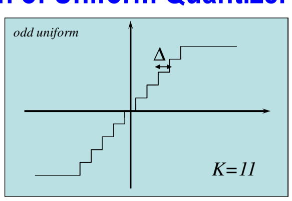
Two-Level Quantizer – (3)



29

Implementation of Uniform Quantizer

* **The quantizer**
  – Odd
  – Uniform
  – K = 11 outp.levels



can be implemented by:

$$Q[x] = y_j = \Delta\,\mathrm{nint}\left(\frac{x}{\Delta}\right) = \Delta\left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor$$

$$j = \mathrm{nint}\left(\frac{x}{\Delta}\right) = \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor$$

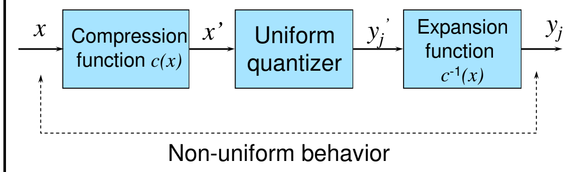
30

5LSE0 Mod 03, Part 3
Non-Uniform Quantization

Non-Uniform Quantizer / 2 Approaches

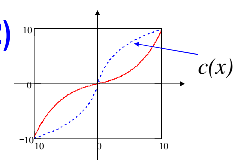
1. **Preprocessing of x by non-linear function followed by uniform quantizer:**
   - Companding (compression-expanding)
   - Simple implementation
   - Popular for audio: logarithmic curves
     - A-law (Europe)
     - and µ-law (USA, Japan)
2. **Lloyd-Max quantizers, minimization of σ_q^2**
   - Complex design
   - More complex implementation than uniform quantizer
   - Additional gain?

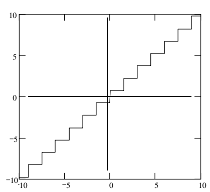
Companding – (1)

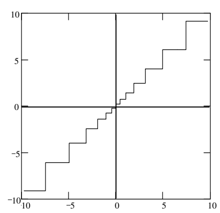


Uniform quantizer is embedded between two non-uniform functions

Companding – (2)







A/µ-law like companding

A- and µ-Law Definition

A-Law

$$c(x) = \begin{cases} x_{\max} \dfrac{a x / x_{\max}}{1+\ln(a)} & 0 \le |x|/x_{\max} \le 1/a \\ x_{\max} \dfrac{\ln(a|x|/x_{\max})}{1+\ln(a)} \operatorname{sgn}(x) & 1/a < |x|/x_{\max} \le 1 \end{cases}$$

µ -Law

$$c(x) = x_{\max} \frac{\ln(1+\mu |x|/x_{\max})}{\ln(1+\mu)} \operatorname{sgn}(x)$$

Lloyd-Max Quantizer – (1)

- **Minimizes quantization noise variance, without enforcing any* structure onto decision thresholds and representation levels**

$$\min \sigma_q^2 = \min \sum_{k=1}^{K} \int_{x_k}^{x_{k+1}} (x-y_k)^2 \, p_X(x)\, \mathrm{d}x$$

for:

$$x_k \qquad k = 2,3,\ldots,K$$

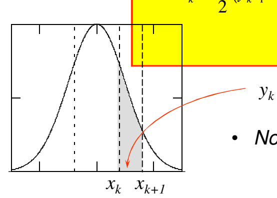
$$y_k \qquad k = 1,2,\ldots,K$$

** except for symmetry of the quantizer*

Lloyd-Max Quantizer – (2)

- **General solution is given by implicit expressions:**

$$x_k = \frac{1}{2}(y_{k-1} + y_k) \qquad y_k = \frac{\int_{x_k}^{x_{k+1}} x p_X(x)\,\mathrm{d}x}{\int_{x_k}^{x_{k+1}} p_X(x)\,\mathrm{d}x}$$



- *Note the structure of the formulas*
  - *Middle of two representation levels*
  - *Weighted average decision levels*

Lloyd-Max Quantizer – (3)

∗ **Property of solution:**
- *if* x_k are known $\Rightarrow y_k$ are known
- *if* y_k are known $\Rightarrow x_k$ are known

∗ **Iterative design necessary**

initial choice, "seed"

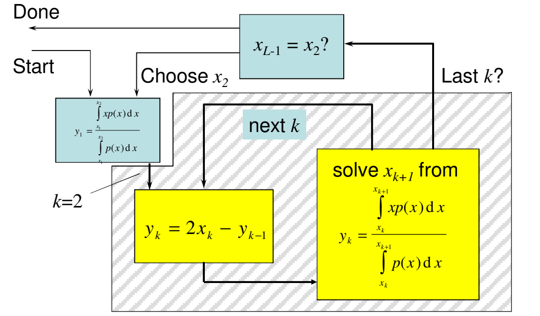
$$x_1 = -\infty$$

$$\text{Assume: } x_2 = g \quad \Rightarrow \quad y_1 = \frac{\int_{x_1}^{x_2} x p(x)\,\mathrm{d}x}{\int_{x_1}^{x_2} p(x)\,\mathrm{d}x} \quad \Rightarrow \quad y_2 = 2x_2 - y_1$$

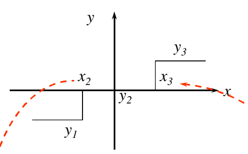
$$\text{Solve } x_3 \text{ from: } \; y_2 = \frac{\int_{x_2}^{x_3} x p(x)\,\mathrm{d}x}{\int_{x_2}^{x_3} p(x)\,\mathrm{d}x} \quad \Rightarrow \quad y_3 = 2x_3 - y_2$$

etcetera

Lloyd-Max / Iterative Solution Scheme



Lloyd-Max / Example iterative Q design



	Iteration			
	1	2	3	4
Assume x_2	-1.000	-0.500	-0.75	-0.612
y_1	-1.525	-1.141	-1.329	-1.224
y_2	-0.475	0.141	-0.171	0.000
y_3	-0.028	0.827	0.368	1.224
x_3	0.419	1.513	0.909	0.612

Examples of Lloyd-Max Quantizers

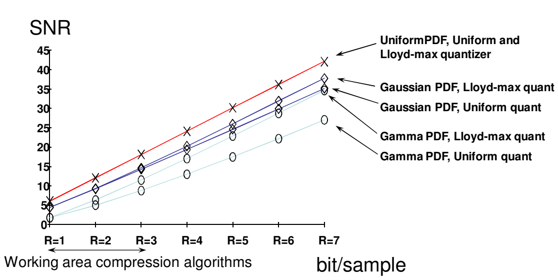
- Example from VcDemo Quantizer

∗ **For *uniform PDF*, Lloyd-Max quantizer is *uniform***

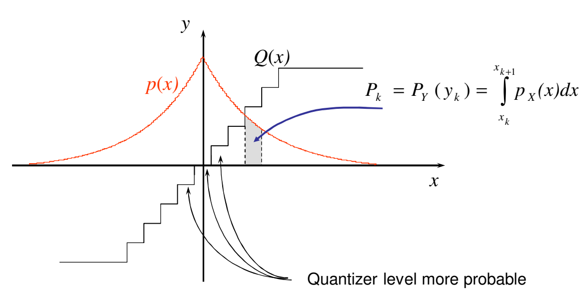
		K=2, 1 *bit/sample*		*K*=4, 2 *bit/sample*		*K*=8, 3 *bit/sample*		*K*=16, 4 *bit/sample*	
PDF	k	x_k	y_k	x_k	y_k	x_k	y_k	x_k	y_k
Gaussian	1	0.000	0.798	0.000	0.453	0.000	0.245	0.000	0.128
	2			0.982	1.510	0.501	0.756	0.258	0.388
	3					1.050	1.344	0.522	0.657
	4					1.748	2.152	0.800	0.942
	5							1.099	1.256
	6							1.437	1.618
	7							1.844	2.069
	8							2.401	2.733
Laplace	1	0.000	0.707	0.000	0.402	0.000	0.233	0.000	0.124
	2			1.127	1.834	0.533	0.833	0.264	0.405
	3					1.253	1.673	0.567	0.729
	4					2.380	3.087	0.920	1.111
	5							1.345	1.578
	6							1.878	2.178
	7							2.597	3.017
	8							3.725	4.432

Uniform quantizer versus Lloyd-Max

For relevant bit rates, Lloyd-Max does often not pay off.

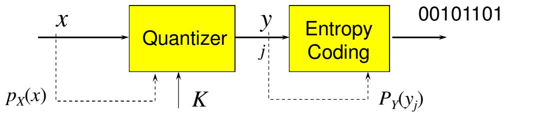


Quantization with Entropy Coding – (1)



- Example from VcDemo Quantizer

Quantization with Entropy Coding – (2)



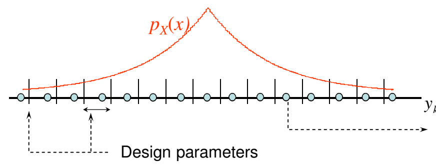
1. Quantizer is optimized for *given* #representation levels K
2. Entropy coder (VLC) is optimized for given probabilities of y_j, yielding average codeword length L

   But, the combination is *not* optimal

   – other quantizer / VLC combinations exist that gives a smaller quantization error variance for same average codeword length

Entropy-Constrained Quantization

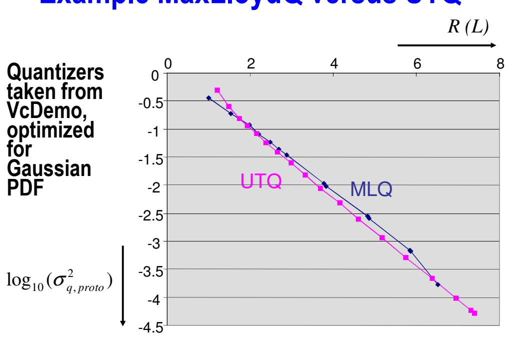
* To find the overall optimal result, quantizer and entropy coder must be *jointly* designed
  – Complex optimization problem
  – Reasonable approximations are obtained by *Uniform Threshold Quantizers (UTQ)*



$$y_k = \frac{\int_{x_k}^{x_{k+1}} x p_X(x)dx}{\int_{x_k}^{x_{k+1}} p_X(x)dx}$$

Example MaxLloydQ versus UTQ

* Quantizers taken from VcDemo, optimized for Gaussian PDF

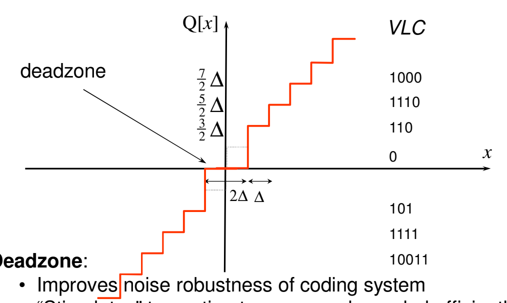


5LSE0 – Mod 03, Part 4
Quantization in practical cases

Quantization in Practice – (1)

* Uniform quantizers are preferred
  – Implementation and limited loss compared to Lloyd-Max
  – Easily scalable (one parameter: step size Δ)
* *Odd* quantizers are often preferred over *even* because of the presence of a representation level at zero
  – In good compression scheme many (near-)zero values occur
  – Zeroes efficiently coded by an entropy coder (run length coding)
* **Audio:** Companding is usual
* **Image/video coding :** No companding
  – Uniform quantizer with *deadzone* is typical

Quantization in Practice – (2)

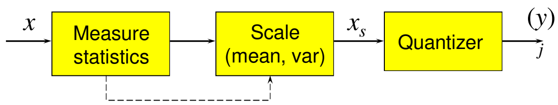


Deadzone:
- Improves noise robustness of coding system
- “Stimulates” truncation to zero: can be coded efficiently

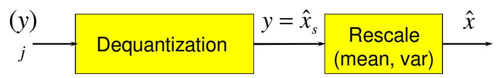
Overall Quantization Error – (1)

* Only *prototype* quantizers are designed with σ_q^2

Sending side



Receiving side



Overall Quantization Error – (2)

* Scaling procedure: $x_s = \dfrac{x - \mu_x}{\sigma_x}$
* Rescaling procedure: $\hat{x} = \sigma_x y + \mu_x = \sigma_x \hat{x}_s + \mu_x$
* Overall effect: Quantization noise variance is scaled with : σ_x^2

$$\sigma_q^2 = E[(\hat{x} - x)^2] = E[(\sigma_x \hat{x}_s - \sigma_x x_s)^2] = \sigma_x^2 E[(\hat{x}_s - x_s)^2]$$
$$= \sigma_x^2 \sigma_{q,\text{prototype}}^2$$

$\Rightarrow$ ***Quantization noise variance is linearly proportional to the variance of the signal to be quantized***

The following sheets were used in previous classes and are included for reference only

Back to MLQuantizers
Back to Prob. Calculation

Quantizer Example - (2)

	Repres. Level	Probability	Codewords (positive)	Codewords (negative)
1	±9.247	0.00007	00010101101010	00010101101011
2	±6.875	0.00031	000101011011	0001010110100
3	±5.519	0.00077	0001010111	00010101100
4	±4/562	0.00152	0001010000	0001010001
5	±3.823	0.00269	000101001	000101010
6	±3.217	0.00444	00011010	00011011
7	±2.703	0.00699	1011010	1011011
8	±2.253	0.01068	0001011	0001100
9	±1.855	0.01590	000111	101100
10	±1.497	0.02318	10111	000100
11	±1.175	0.03316	01000	01001
12	±0.884	0.04658	1100	1101
13	±0.622	0.06441	0101	1010
14	±0.388	0.08797	111	0000
15	±0.180	0.11987	011	100
16	0.000	0.16292	001	

* Simple binary coding would require $\lceil \log_2(31) \rceil$ =5 bits
* $H(S)$ = 3.876 bit/repr.level
* L = 3.912 bit/repr.level

Back to Quantizers

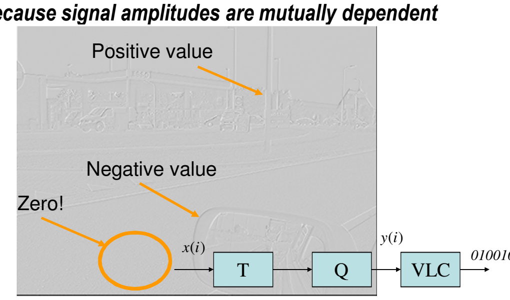
Application to Signal Coding

* **We will see that efficient "transforms" used in compression produce**
  – A lot of "zero" values
  – Some (significant) non-zero values
* **Typical symbol sequences to be coded:**
  "5 1 0 0 0 0 0 0 0 3 0 0 6 0 0 0 0 1 0 0 0 0 …"
  – Will be done by {zero-run, non-zero symbol} pairs
  – Here: "{0,5}, {0,1}, {7,3}, {2,6}, {4,1}, …"
  – The pairs will now be assigned a Huffman code

Back to Quantizers

Why can Signals be Compressed?

Because signal amplitudes are mutually dependent



The following sheets are included for further reference

Back to Quantization Noise

Note on Quantization Noise Variance

* We wish to calculate the variance of *q*:

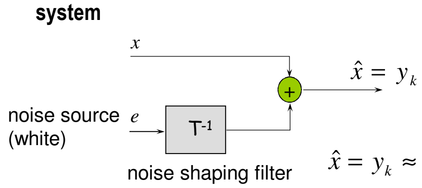
$$\sigma_q^2 = E[q^2] - \underbrace{E[q]^2}_{=0} = \int_{-\infty}^{\infty} q^2 p_q(q) dq$$

* Try to calculate $p_q(q)$, but with non-linear relation of $q=x-Q[x]$, this is not easy
* Much easier, take following short-cut (E-operator):

$$\sigma_q^2 = E[q^2] = E[(x - Q[x])^2]$$
$$= \int_{-\infty}^{\infty} (x - Q[x])^2 \, p_X(x) \, \mathrm{d}x$$

Modeling of Quantizers

* Quantizer is often the only non-linear element in a compression system
* Linear (approximating) models are sometimes used to analyze the performance of a quantizer / compression system



$$\hat{x} = y_k \approx x + T^{-1}e$$