Multimedia Video Coding & Architectures (5LSE0), Module 04

Rate-distortion Theory and
and Performance of Scalar Quantization

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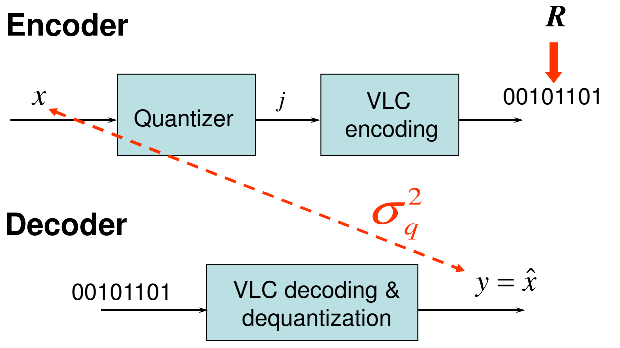
slides version 1.0

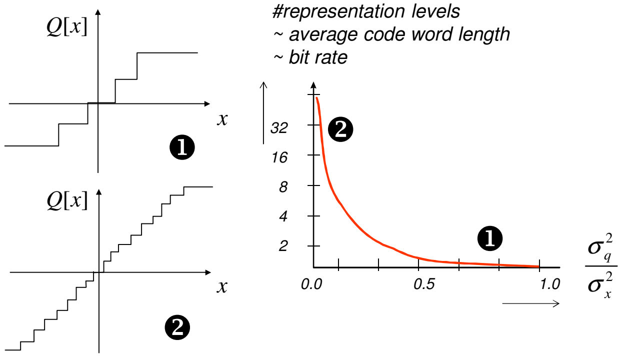
5LSE0 - Mod 04

Starting considerations and motivation

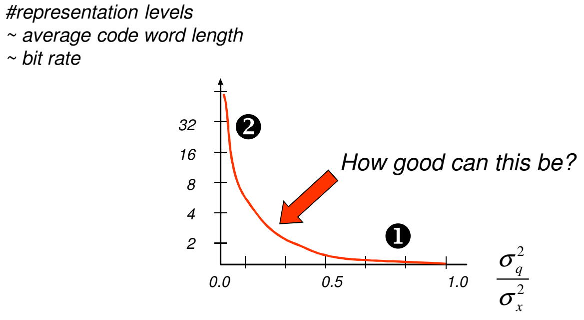
Quantizer Design Problem



Bit Rate versus Distortion – (1)

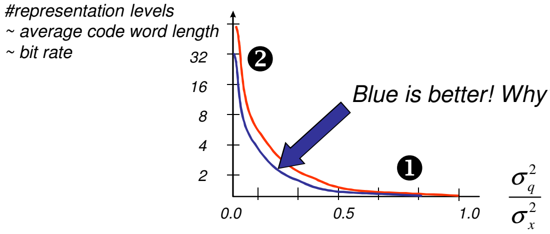

#representation levels
~ average code word length
~ bit rate

Bit Rate versus Distortion – (2)



How good can this be?

Bit Rate versus Distortion – (3)

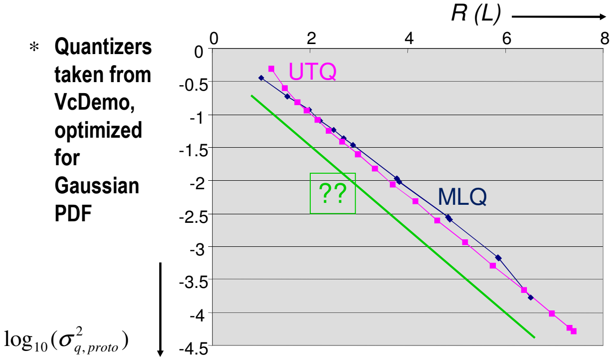


Blue is better! Why?

1. *What about the optimal trade-off between rate and distortion from a theoretical point of view?*
2. *How do scalar quantizers perform relative to the optimum?*

Remember Performance Example…

- **Quantizers taken from VcDemo, optimized for Gaussian PDF**



Why can Signals be Compressed?

Because infinite accuracy of signal amplitudes is (perceptually) irrelevant

Question 1:

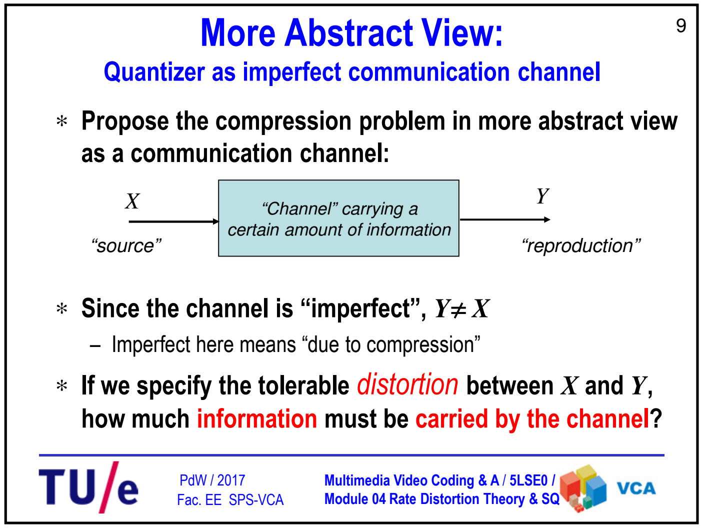
What is the *best possible* trade-off between required bit rate and resulting distortion?

(Rate-Distortion Theory)

Question 2:

How do we *implement* a system that gives us that best possible trade-off?

(Scalar and Vector Quantization Theory)



More Abstract View:

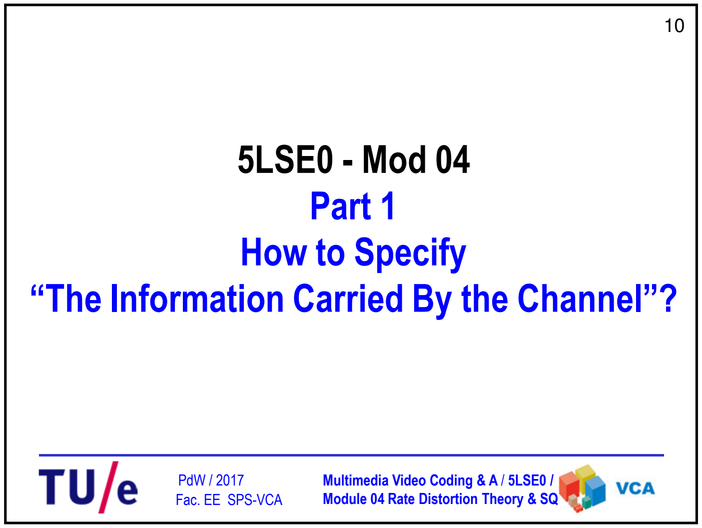
Quantizer as imperfect communication channel

- Propose the compression problem in more abstract view as a communication channel:

X → "Channel" carrying a certain amount of information → Y

"source" "reproduction"

- Since the channel is "imperfect", $Y \neq X$
  - Imperfect here means "due to compression"
- If we specify the tolerable *distortion* between X and Y, how much **information** must be **carried by the channel**?



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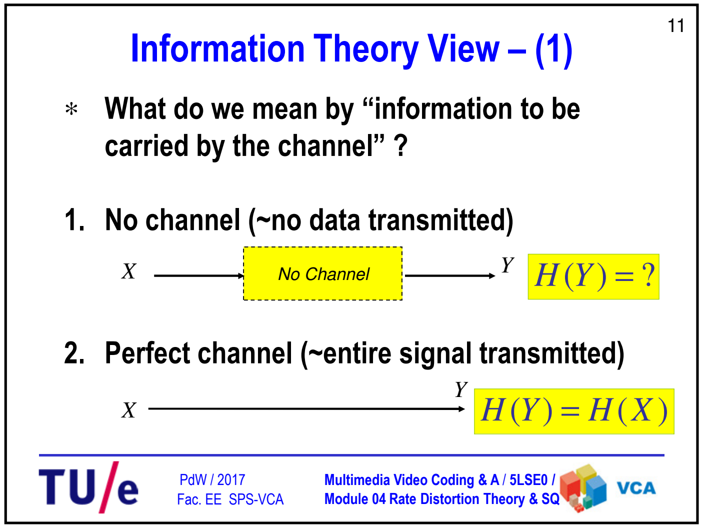
Part 1

How to Specify "The Information Carried By the Channel"?



Information Theory View – (1)

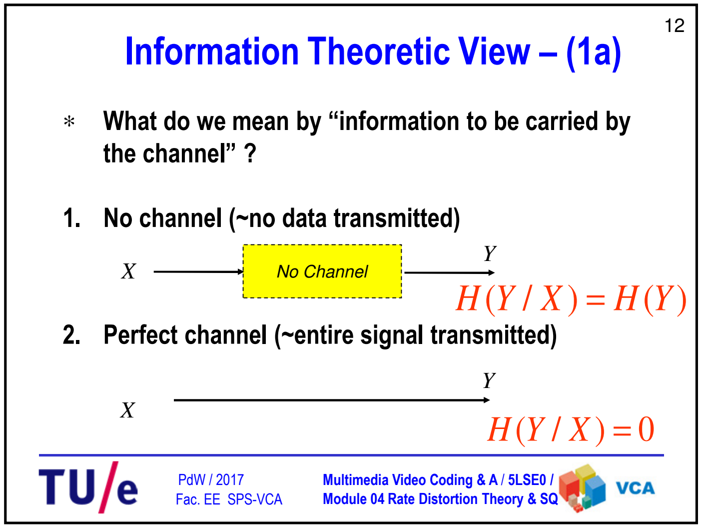
- What do we mean by "information to be carried by the channel" ?

1. No channel (~no data transmitted)

X → No Channel → Y $H(Y) = ?$

2. Perfect channel (~entire signal transmitted)

X → Y $H(Y) = H(X)$



Information Theoretic View – (1a)

- What do we mean by "information to be carried by the channel" ?

1. No channel (~no data transmitted)

X → No Channel → Y

$$H(Y/X) = H(Y)$$

2. Perfect channel (~entire signal transmitted)

X → Y

$$H(Y/X) = 0$$

Conditional Information $H(Y/X)$ – (1)

- **Start with entropy $H(Y)$**
- **After data transmission over the channel, we consider the remaining entropy $H(Y/X)$**
- **Conditional Information $H(Y/X)$**
  - *Amount of uncertainty remaining* in Y *after observing* X
  - Sort of “obvious”:
    - *a.* X and Y (statistically) independent : $H(Y/X) = H(Y)$
    - *b.* X and Y are identical : $H(Y/X) = 0$
- **For discrete sources:** $0 \le H(Y/X) \le H(Y)$

Conditional Information $H(Y/X)$ – (2)

- **Definition of Conditional Information for Memoryless Source:**

$$H(Y/X) = -\sum_{i=1}^{N}\sum_{j=1}^{M} P_{XY}(x_i, y_j)\log_2[Q_{Y|X}(y_j/x_i)]$$

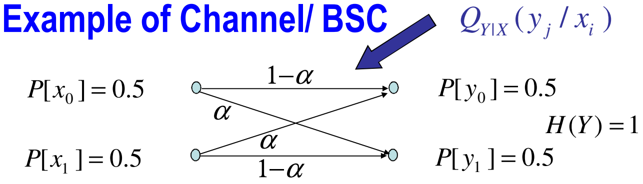
$$H(Y/X) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{p_{XY}(x,y)}_{}\log_2[\underbrace{q_{Y|X}(y/x)}_{}]\,\mathrm{d}x\,\mathrm{d}y$$

Joint pdf/pmf

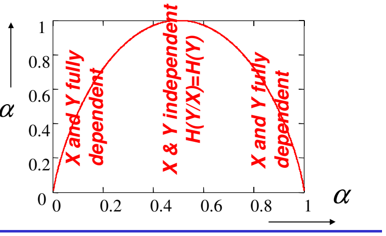
conditional pdf/pmf

Represents “quantization” in compression context, hence “q”

Example of Channel/ BSC

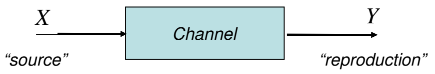


$H(Y/X) = \text{function of } \alpha$



Information Theoretic View – (2)

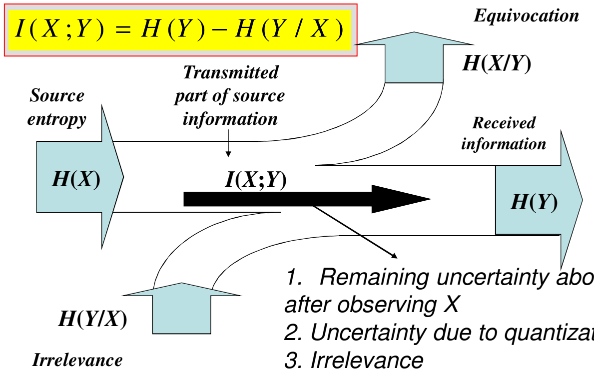
- **We are not so much interested in $H(Y/X)$ itself, but**
  - the ***change*** in the amount information in Y, between
    - the case of having no channel
    - the case of having a channel
  - because the ***difference*** is apparently the information transmitted over the channel



- **Difference is *mutual information* $I(X;Y)$, or bit rate R**

Mutual Information Defined – (1)

$I(X;Y) = H(Y) - H(Y/X)$



Equivocation

$H(X/Y)$

Source entropy

Transmitted part of source information

Received information

$H(X)$

$I(X;Y)$

$H(Y)$

$H(Y/X)$

Irrelevance

1. *Remaining uncertainty about Y after observing X*
2. *Uncertainty due to quantization*
3. *Irrelevance*

Mutual Information $I(X;Y)$ – (2)

- Substituting definitions for $H(Y)$ and $H(Y/X)$:

$$I(X;Y)=\sum_{i=1}^{N}\sum_{j=1}^{M}P_{XY}(x_i,y_j)\log_2\left[\frac{P_{XY}(x_i,y_j)}{P_X(x_i)P_Y(y_j)}\right]$$

$$I(X;Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p_{XY}(x,y)\log_2\left[\frac{p_{XY}(x,y)}{p_X(x)p_Y(y)}\right]\mathrm{d}\,x\,\mathrm{d}\,y$$

(Definitions hold for memoryless sources)

Mutual Information $I(X;Y)$ – (3)

- **Interpretations of** $I(X;Y)=H(Y)-H(Y/X)$

1. **If X and Y are independent, then knowledge about X**
   - does not decrease the uncertainty about Y
   - does not change the information obtained when observing an outcome of X

   in other words

   $$H(Y/X)=H(Y)$$

   $$I(X;Y)=H(Y)-H(Y/X)=H(Y)-H(Y)=0$$

   Conclusion: No information transmitted

   Case: No channel

Mutual Information $I(X;Y)$ – (4)

- **Interpretations of** $I(X;Y)=H(Y)-H(Y/X)$

2. If X and Y are equal, then knowledge about X
   - does not leave any uncertainty about Y
   - gives us all the information there is, even without observing an outcome of X

   in other words

   $$H(Y/X)=0$$

   $$I(X;Y)=H(Y)-H(Y/X)=H(Y)-0=H(Y)$$

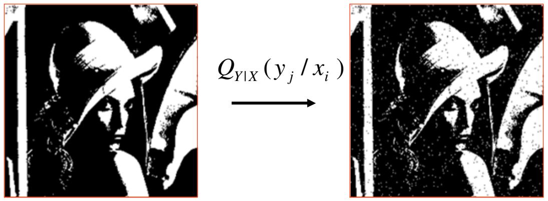
   Conclusion: Information of source has been transmitted

   Case: Perfect channel

Mutual Inform./ Binary Example – (1)

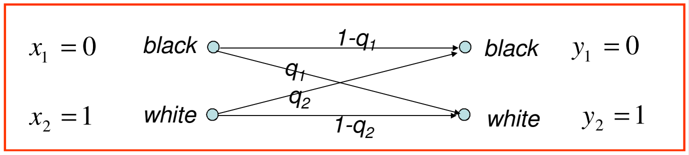
$$I(X;Y) = H(Y) - H(Y/X)$$

- Example binary image



Assume P("white") = 0.5 *Assume amplitude "white" = 1*

Mutual Inform./ Binary Example – (2)



$$H(Y) = -P_Y(y_1)\log_2 P_Y(y_1) - P_Y(y_2)\log_2 P_Y(y_2)$$

$$= -\frac{(1-q_1+q_2)}{2}\log_2\frac{(1-q_1+q_2)}{2} - \frac{(1+q_1-q_2)}{2}\log_2\frac{(1+q_1-q_2)}{2}$$

$$H(Y/X) = -\frac{1-q_1}{2}\log_2(1-q_1) - \frac{q_1}{2}\log_2(q_1) - \frac{q_2}{2}\log_2(q_2) - \frac{1-q_2}{2}\log_2(1-q_2)$$

Memoryless source: pixels are assumed to be independent

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Part 2
How to Specify
" Distortion between X and Y "?

Distortion Measures – (1)

- **Send amplitude/source value x**
- **Reproduce/reconstruct amplitude value y**
- **Distortion defined as $\rho(x,y)$**

- **In audio/image/video compression, usually**
  - $\rho(x,y) = (x-y)^2$ (mean square error, SNR)
  - a perceptual variant of this quadratic measure
    - (to be discussed later in course)

Distortion Measures – (2)

- **Average distortion**
  - Expected value of the distortion
  - (or) Weighted average over all possible (x,y) combinations
- **Start with the Discrete Case:**

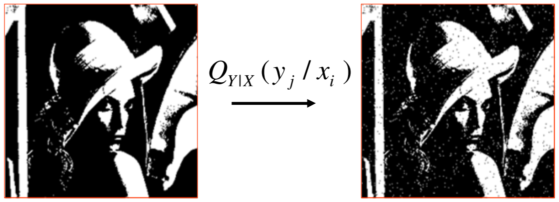
$$D = \sum_{i=1}^{N}\sum_{j=1}^{M} \rho(x_i, y_j) P_{XY}(x_i, y_j)$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{M} \rho(x_i, y_j) Q_{Y|X}(y_j / x_i) P_X(x_i)$$

Distortion / Binary Example – (3)

$$D == \sum_{i=1}^{N}\sum_{j=1}^{M} \rho(x_i, y_j) Q_{Y|X}(y_j / x_i) P_X(x_i)$$

Represents channel *Given and fixed*

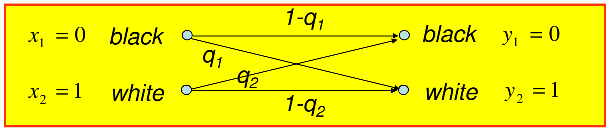
- **Example**



$Q_{Y|X}(y_j / x_i)$

Distortion / Binary Example – (4)

$$D == \sum_{i=1}^{N}\sum_{j=1}^{M} \rho(x_i, y_j) Q_{Y|X}(y_j / x_i) P_X(x_i)$$



$$D = (x_1 - y_1)^2 (1 - q_1) P_X(x_1) + (x_1 - y_2)^2 q_1 P_X(x_1)$$
$$+ (x_2 - y_1)^2 q_2 (1 - P_X(x_1)) + (x_2 - y_2)^2 (1 - q_2)(1 - P_X(x_1))$$
$$= (-1)^2 q_1 \frac{1}{2} + (1)^2 q_2 \frac{1}{2} = \frac{q_1 + q_2}{2}$$

Distortion Measures – (5)

- **For the continuous case, we define**

$$D = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x, y) p_{XY}(x, y) \, \mathrm{d}x \, \mathrm{d}y$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x, y) q_{Y|X}(y / x) p_X(x) \, \mathrm{d}x \, \mathrm{d}y$$

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Part 3

Rate-Distortion Lower Bound: Combining $I(X;Y)$ and D

Rate-Distortion Theory – (1)

- **For the abstract communication channel:**

We now know:

1. How to calculate the "Rate"

$$I(X;Y) = H(Y) - H(Y/X)$$

1. How to calculate the distortion D
2. That both quantities depend on $Q_{Y|X}(y,x)$
   - D_Q and $I_Q(X;Y)$

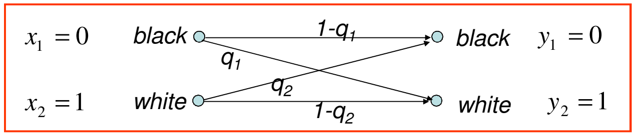
Rate-Distortion Theory – (2)

- **Problem in Rate-Distortion Theory:**
  - Minimize the rate (mutual information) needed to reproduce X at the receiving side with a distortion smaller than some value $D*$
  - The minimization is done over the parameter that describes the communication channel (our case: the compression system), namely $Q_{Y|X}(y,x)$

$$\min_{Q_{Y|X}(y/x)} \quad I_Q(Y;X) \qquad \text{given } D_Q \le D^*$$

  - Result is known as **Rate-Distortion bound** $R(D)$

Rate Distortion / Binary Example –(1)

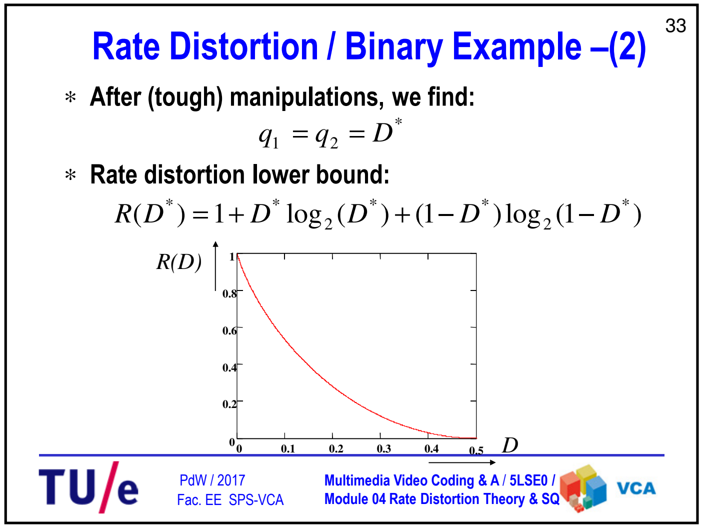


$$\min_{\boxed{q_1, q_2}} I(X;Y) = H(Y) - H(Y/X)$$

$$= -\frac{(1-q_1+q_2)}{2}\log_2\frac{(1-q_1+q_2)}{2} - \frac{(1+q_1-q_2)}{2}\log_2\frac{(1+q_1-q_2)}{2}$$

$$+\frac{1-q_1}{2}\log_2(1-q_1) + \frac{q_1}{2}\log_2(q_1) + \frac{q_2}{2}\log_2(q_2) + \frac{1-q_2}{2}\log_2(1-q_2)$$

Given $\quad : D = \frac{q_1+q_2}{2} \le D^*$



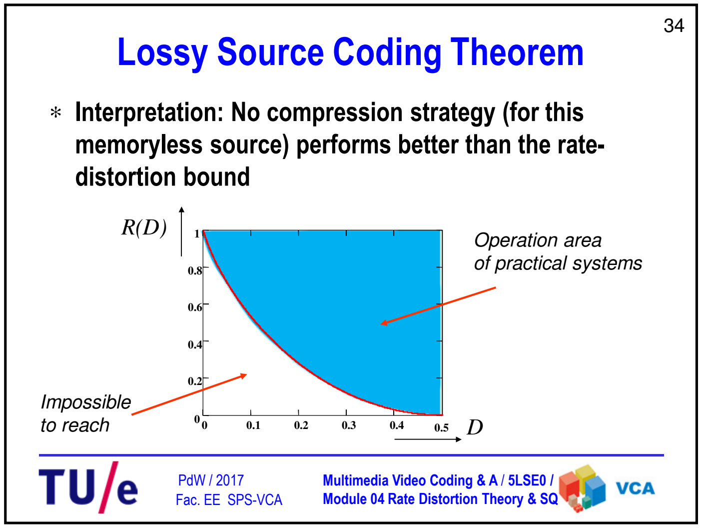
Rate Distortion / Binary Example –(2)

- After (tough) manipulations, we find:

$$q_1 = q_2 = D^*$$

- Rate distortion lower bound:

$$R(D^*) = 1 + D^* \log_2(D^*) + (1-D^*)\log_2(1-D^*)$$

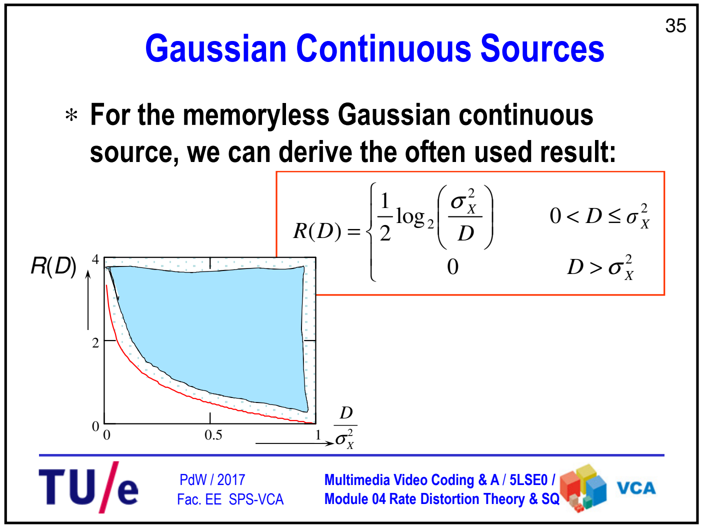


Lossy Source Coding Theorem

- Interpretation: No compression strategy (for this memoryless source) performs better than the rate-distortion bound

Operation area of practical systems

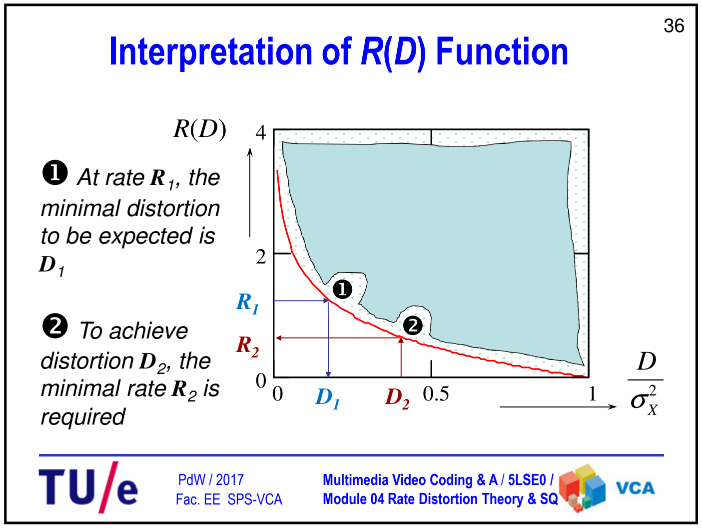
Impossible to reach



Gaussian Continuous Sources

- For the memoryless Gaussian continuous source, we can derive the often used result:

$$R(D) = \begin{cases} \frac{1}{2}\log_2\left(\frac{\sigma_X^2}{D}\right) & 0 < D \le \sigma_X^2 \\ 0 & D > \sigma_X^2 \end{cases}$$



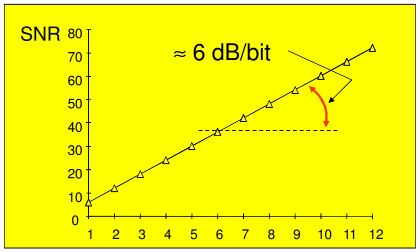
Interpretation of *R*(*D*) Function

❶ *At rate R_1, the minimal distortion to be expected is D_1*

❷ *To achieve distortion D_2, the minimal rate R_2 is required*

R(D) Function as SNR Function

$$\begin{cases} R = \frac{1}{2}\log_2\left(\frac{\sigma_X^2}{D}\right) = \frac{1}{2}\log_2\left(\frac{\sigma_X^2}{\sigma_q^2}\right) \\ SNR = 10\log_{10}\left(\frac{\sigma_X^2}{\sigma_q^2}\right) \end{cases} \Rightarrow \quad SNR = 6.02R$$



Remarks on Rate Distortion *R(D)*

- Rate-distortion bound is most relevant for continuous sources as these can never be reconstructed perfectly
- Although multimedia data is digitized (~large number of discrete amplitudes), theory of continuous sources is virtually always used
- Rate-distortion theory does not tell us *how* to realize a system that achieves the bound (~best possible compression)
- There are few cases for which *R*(*D*) can be calculated
- *R*(*D*) is a convex function $0 \le R(D) \le H(X)$

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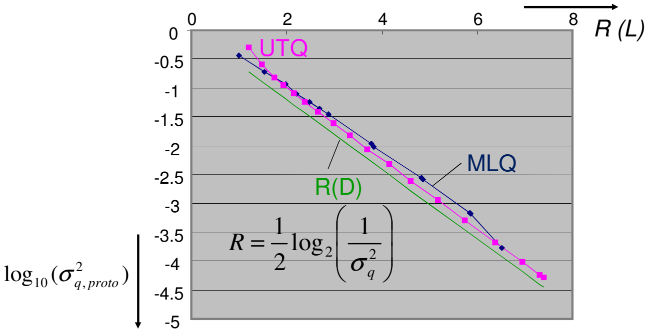
How Well Do Scalar Quantizers Approach the Rate-Distortion Bound?

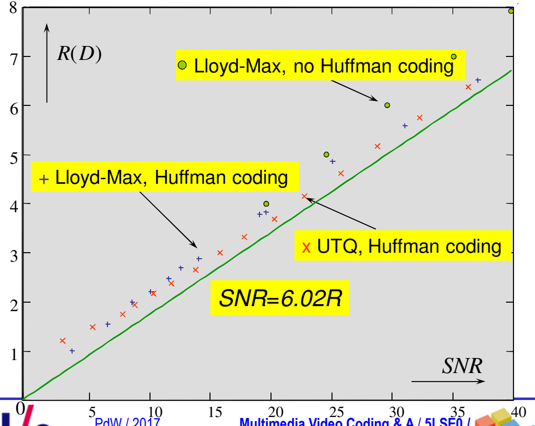
Comparison w. Designed Quantizers – (1)



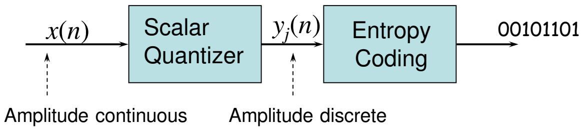
$$R = \frac{1}{2}\log_2\left(\frac{1}{\sigma_q^2}\right)$$

$\log_{10}(\sigma^2_{q,proto})$

Comparison w. designed Quantizers – (2)



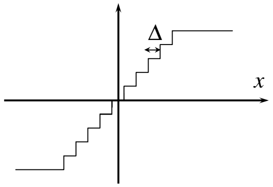
Can We Explain the “Gap”?

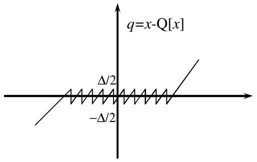


- **The above scheme is a *practical realization* of a compression system**
- **What about the rate-distortion behavior?**
  - What is the entropy of the quantizer outputs?
  - How is this related to the behavior of the quantizer?

Approximate Quantizer Distortion – (1)

- **Approximate behavior of a uniform quantizer (at *high bit rates):* $p_q(q)$ ~ uniform pdf**





$$p(q) = p(x - Q[x]) \approx \begin{cases} \dfrac{1}{\Delta} & -\dfrac{\Delta}{2} \le q \le \dfrac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Approximate Quantizer Distortion – (2)

- **For *Lloyd-Max* and *Uniform Threshold Quantizers* this approximation also holds “reasonably well”**
- **Distortion D for given Δ is now approximately equal to:**

$$\boxed{D = \sigma_q^2 = \frac{1}{12}\Delta^2}$$

R	Δ (σ_x=1)			SNR (dB)=$10\log_{10}(1/\ \sigma_q^2)$		
	Uniform	Gaussian	Laplace	Uniform	Gaussian	Laplace
4	0.217	0.335	0.461	24.08 (*24.06*)	19.38 (*20.29*)	15.96 (*17.51*)
5	0.108	0.188	0.280	30.10 (*30.12*)	24.57 (*25.30*)	20.60 (*21.84*)
6	0.054	0.104	0.166	36.12 (*36.14*)	29.83 (*30.45*)	25.36 (*26.38*)
7	0.027	0.057	0.096	42.14 (*42.16*)	35.13 (*35.67*)	30.23 (*31.14*)
8	0.013	0.031	0.055	48.17 (*48.51*)	40.34 (*40.96*)	35.14 (*35.98*)

Entropy of Quantizer Output Levels

- **For the quantizer representation levels, it holds**

$$P(Y=y_j)=\int_{x_j}^{x_{j+1}} p(x)\,\mathrm{d}x \approx \Delta p(x=y_j)$$

- **This leads to the following approximation for the entropy of the quantizer outputs:**

$$H(Y)=-\sum_{j=1}^{N} P[Y=y_j]\log_2 P[Y=y_j]$$

$$\approx H(X)-\log_2(\Delta)$$

Entropy of Quantizer Output Levels (2)

- **Entropy of quantizer outputs, i.e. the representation levels or quantized values):**

$$H(Y)=H(X)-\log_2(\Delta)$$

(Resulting) entropy of quantized signal

(Given) entropy of signal to be coded

Influence of quantizer accuracy

- ***More accurate*** quantization generates symbols with ***more entropy***.
- Similar results can be obtained for other quantizer types with sufficiently smooth PDFs and at high bit rates

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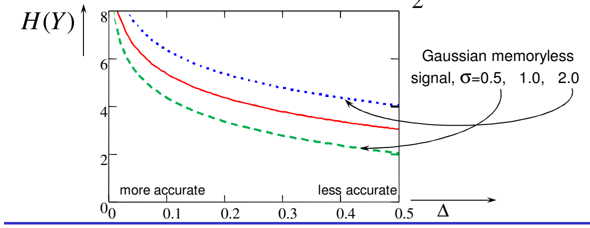
Application to Gaussian Source

- **We have** : $H(Y)=H(X)-\log_2(\Delta)$

and for Gaussian sources : $H(X)=\frac{1}{2}\log_2(2\pi e)+\log_2(\sigma_x)$

yielding : $H(Y)=\frac{1}{2}\log_2(2\pi e)+\log_2(\frac{\sigma_x}{\Delta})$



Gaussian memoryless signal, σ=0.5, 1.0, 2.0

Compare to Rate-Distortion Function – (1)

- **For a Gaussian memoryless signal, the quantizer with step size Δ generates:**
  - Symbols with entropy: $H(Y)=\frac{1}{2}\log_2(2\pi e)+\log_2(\frac{\sigma_x}{\Delta})$
  - Distortion: $D=\sigma_q^2=\frac{1}{12}\Delta^2$
- **Combining these two gives:**

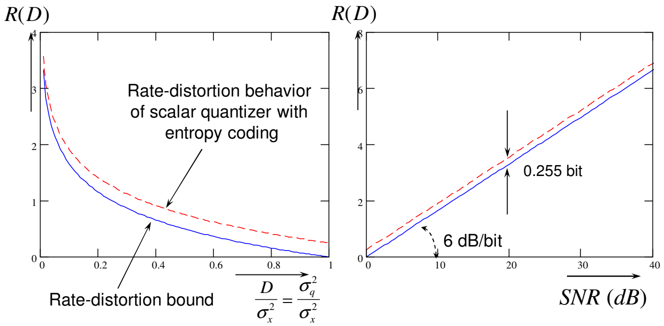
$$H(Y)\approx\frac{1}{2}\log_2(\frac{\sigma_X^2}{D})+\frac{1}{2}\log_2(\frac{e\pi}{6})$$

$$=R(D)+\frac{1}{2}\log_2(\frac{e\pi}{6})$$

Rate-distortion bound

the Gap

Compare to Rate-Distortion Function – (2)



The gap is known as the Gish-Pierce asymptote

Exceed Gish-Pierce Asymptote?

- **Quantize multiple amplitudes *at the same time***

Vector Quantization (VQ)

- **The rate-distortion bound can be approximated arbitrarily closely by taking larger vector lengths N**

- **VQ is a special topic, rarely used in practice ….**