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Multimedia Video Coding & Architectures (5LSE0), Module 04

Rate-distortion Theory and and Performance of Scalar Quantization

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slides version 1.0

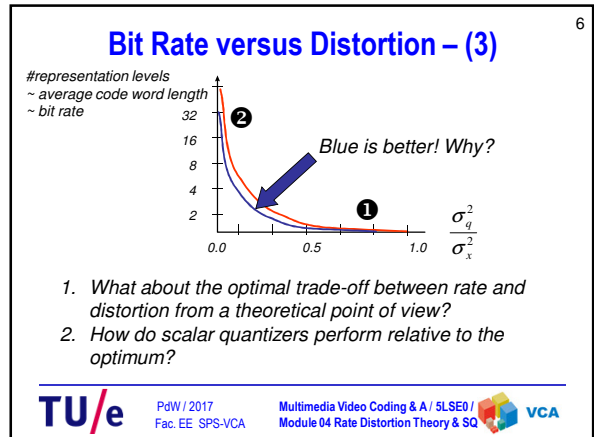
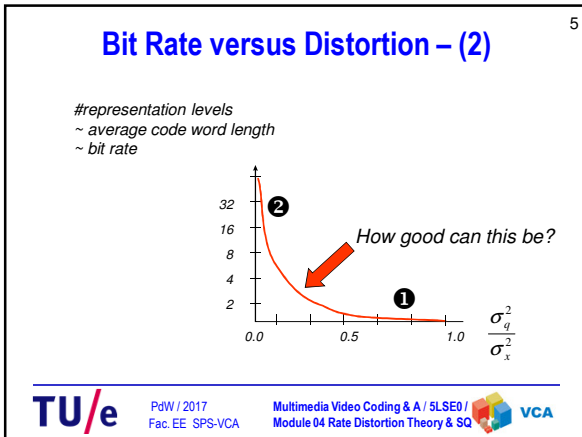
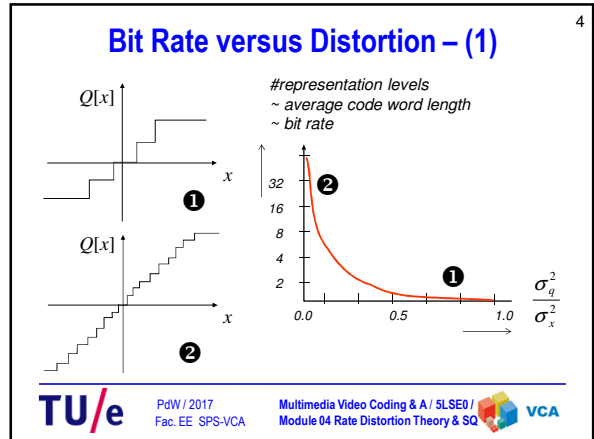
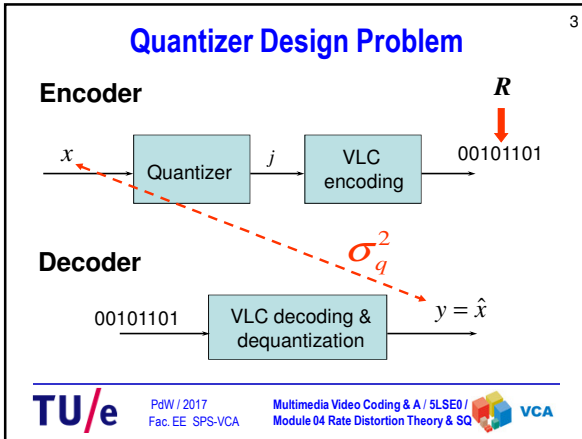
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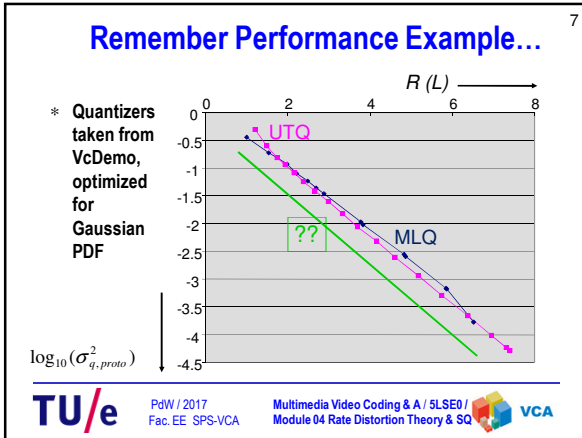
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Starting considerations and motivation

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Why can Signals be Compressed?

Because infinite accuracy of signal amplitudes is (perceptually) irrelevant

Question 1:
What is the **best possible trade-off** between required bit rate and resulting distortion?
(Rate-Distortion Theory)

Question 2:
How do we **implement** a system that gives us that best possible trade-off?
(Scalar and Vector Quantization Theory)

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More Abstract View:

Quantizer as imperfect communication channel

* Propose the compression problem in more abstract view as a communication channel:

* Since the channel is "imperfect", $Y \neq X$
– Imperfect here means "due to compression"

* If we specify the tolerable *distortion* between X and Y , how much **information** must be **carried by the channel**?

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Part 1

How to Specify "The Information Carried By the Channel"?

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Information Theory View – (1)

* What do we mean by "information to be carried by the channel" ?

- No channel (~no data transmitted)
- Perfect channel (~entire signal transmitted)

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Information Theoretic View – (1a)

* What do we mean by "information to be carried by the channel" ?

- No channel (~no data transmitted)
- Perfect channel (~entire signal transmitted)

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Conditional Information $H(Y/X) - (1)$ 13

- * Start with entropy $H(Y)$
- * After data transmission over the channel, we consider the remaining entropy $H(Y/X)$
- * **Conditional Information $H(Y/X)$**
 - Amount of uncertainty remaining in Y after observing X
 - Sort of "obvious":
 - X and Y (statistically) independent : $H(Y/X) = H(Y)$
 - X and Y are identical : $H(Y/X) = 0$
- * **For discrete sources:** $0 \leq H(Y/X) \leq H(Y)$

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Conditional Information $H(Y/X) - (2)$ 14

- * **Definition of Conditional Information for Memoryless Source:**

$$H(Y/X) = - \sum_{i=1}^N \sum_{j=1}^M P_{XY}(x_i, y_j) \log_2 [Q_{Y|X}(y_j / x_i)]$$

$$H(Y/X) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{XY}(x, y) \log_2 [q_{Y|X}(y/x)] dx dy$$

Joint pdf/pmf
conditional pdf/pmf

Represents "quantization" in compression context, hence "q"

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Example of Channel/ BSC 15

$P[x_0] = 0.5$

$P[x_1] = 0.5$

$P[y_0] = 0.5$

$P[y_1] = 0.5$

$H(Y) = 1$

$H(Y/X) = \text{function of } \alpha$

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Information Theoretic View - (2) 16

- * **We are not so much interested in $H(Y/X)$ itself, but**
 - the **change** in the amount information in Y , between
 - the case of having no channel
 - the case of having a channel
 - because the **difference** is apparently the information transmitted over the channel

- * **Difference is *mutual information* $I(X;Y)$, or bit rate R**

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Mutual Information Defined - (1) 17

$I(X;Y) = H(Y) - H(Y/X)$

Source entropy

$H(X)$

Transmitted part of source information

$I(X;Y)$

Received information

$H(Y)$

Equivocation

$H(X|Y)$

Irrelevance

$H(Y/X)$

1. Remaining uncertainty about Y after observing X
2. Uncertainty due to quantization
3. Irrelevance

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Mutual Information $I(X;Y) - (2)$ 18

- * **Substituting definitions for $H(Y)$ and $H(Y/X)$:**

$$I(X;Y) = \sum_{i=1}^N \sum_{j=1}^M P_{XY}(x_i, y_j) \log_2 \left[\frac{P_{XY}(x_i, y_j)}{P_X(x_i) P_Y(y_j)} \right]$$

$$I(X;Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{XY}(x, y) \log_2 \left[\frac{p_{XY}(x, y)}{p_X(x) p_Y(y)} \right] dx dy$$

(Definitions hold for memoryless sources)

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Mutual Information $I(X;Y)$ – (3)

* Interpretations of $I(X;Y) = H(Y) - H(Y|X)$

- If X and Y are independent, then knowledge about X
 - does not decrease the uncertainty about Y
 - does not change the information obtained when observing an outcome of X

in other words

$$H(Y|X) = H(Y)$$

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Y) = 0$$

Conclusion: No information transmitted
Case: No channel

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Mutual Information $I(X;Y)$ – (4)

* Interpretations of $I(X;Y) = H(Y) - H(Y|X)$

- If X and Y are equal, then knowledge about X
 - does not leave any uncertainty about Y
 - gives us all the information there is, even without observing an outcome of X

in other words

$$H(Y|X) = 0$$

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - 0 = H(Y)$$

Conclusion: Information of source has been transmitted
Case: Perfect channel

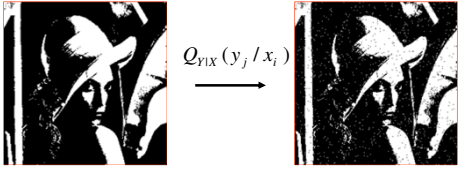
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Mutual Inform./ Binary Example – (1)

$$I(X;Y) = H(Y) - H(Y|X)$$

* Example binary image

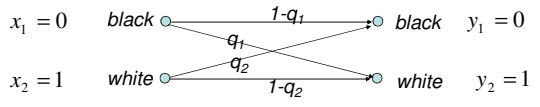


Assume $P(\text{"white"}) = 0.5$ Assume amplitude "white" = 1

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Mutual Inform./ Binary Example – (2)



$$H(Y) = -P_Y(y_1) \log_2 P_Y(y_1) - P_Y(y_2) \log_2 P_Y(y_2)$$

$$= -\frac{(1-q_1+q_2)}{2} \log_2 \frac{(1-q_1+q_2)}{2} - \frac{(1+q_1-q_2)}{2} \log_2 \frac{(1+q_1-q_2)}{2}$$

$$H(Y|X) = -\frac{1-q_1}{2} \log_2(1-q_1) - \frac{q_1}{2} \log_2(q_1) - \frac{q_2}{2} \log_2(q_2) - \frac{1-q_2}{2} \log_2(1-q_2)$$

Memoryless source: pixels are assumed to be independent

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5LSE0 - Mod 04 Part 2 How to Specify "Distortion between X and Y "?

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Distortion Measures – (1)

- * Send amplitude/source value x
- * Reproduce/reconstruct amplitude value y
- * Distortion defined as $\rho(x,y)$

- * In audio/image/video compression, usually
 - $\rho(x,y) = (x-y)^2$ (mean square error, SNR)
 - a perceptual variant of this quadratic measure
 - (to be discussed later in course)

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Distortion Measures – (2)

- * **Average distortion**
 - Expected value of the distortion
 - (or) Weighted average over all possible (x_i, y_j) combinations
- * **Start with the Discrete Case:**

$$D = \sum_{i=1}^N \sum_{j=1}^M \rho(x_i, y_j) P_{XY}(x_i, y_j)$$

$$= \sum_{i=1}^N \sum_{j=1}^M \rho(x_i, y_j) Q_{Y|X}(y_j / x_i) P_X(x_i)$$

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Distortion / Binary Example – (3)

$$D = \sum_{i=1}^N \sum_{j=1}^M \rho(x_i, y_j) Q_{Y|X}(y_j / x_i) P_X(x_i)$$

Represents channel
Given and fixed

- * **Example**

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Distortion / Binary Example – (4)

$$D = \sum_{i=1}^N \sum_{j=1}^M \rho(x_i, y_j) Q_{Y|X}(y_j / x_i) P_X(x_i)$$

$x_1 = 0$	black	$\xrightarrow{1-q_1}$	black	$y_1 = 0$
		$\xrightarrow{q_1}$	white	
$x_2 = 1$	white	$\xrightarrow{q_2}$	black	$y_2 = 1$
		$\xrightarrow{1-q_2}$	white	

$$D = (x_1 - y_1)^2 (1 - q_1) P_X(x_1) + (x_1 - y_2)^2 q_1 P_X(x_1) + (x_2 - y_1)^2 q_2 (1 - P_X(x_1)) + (x_2 - y_2)^2 (1 - q_2) (1 - P_X(x_1))$$

$$= (-1)^2 q_1 \frac{1}{2} + (1)^2 q_2 \frac{1}{2} = \frac{q_1 + q_2}{2}$$

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Distortion Measures – (5)

- * **For the continuous case, we define**

$$D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y) p_{XY}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y) q_{Y|X}(y / x) p_X(x) dx dy$$

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5LSE0 – Mod 04 Part 3

Rate-Distortion Lower Bound: Combining $I(X;Y)$ and D

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Rate-Distortion Theory – (1)

- * **For the abstract communication channel:**

We now know:

- How to calculate the "Rate"

$$I(X;Y) = H(Y) - H(Y / X)$$
- How to calculate the distortion D
- That both quantities depend on $Q_{Y|X}(y, x)$
 - D_Q and $I_Q(X;Y)$

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Rate-Distortion Theory – (2)

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* **Problem in Rate-Distortion Theory:**

- Minimize the rate (mutual information) needed to reproduce X at the receiving side with a distortion smaller than some value D^*
- The minimization is done over the parameter that describes the communication channel (our case: the compression system), namely $Q_{Y|X}(y,x)$

$$\min_{Q_{Y|X}(y,x)} I_Q(Y; X) \quad \text{given } D_Q \leq D^*$$

Result is known as **Rate-Distortion bound** $R(D)$

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Rate Distortion / Binary Example –(1)

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$x_1 = 0$ black $\xrightarrow{1-q_1}$ black $y_1 = 0$
 $\quad \quad \quad \searrow^{q_1}$

$x_2 = 1$ white $\xrightarrow{1-q_2}$ white $y_2 = 1$
 $\quad \quad \quad \searrow^{q_2}$

$\min_{q_1, q_2} I(X; Y) = H(Y) - H(Y|X)$
 $= -\frac{(1-q_1+q_2)}{2} \log_2 \frac{(1-q_1+q_2)}{2} - \frac{(1+q_1-q_2)}{2} \log_2 \frac{(1+q_1-q_2)}{2}$
 $+ \frac{1-q_1}{2} \log_2(1-q_1) + \frac{q_1}{2} \log_2(q_1) + \frac{q_2}{2} \log_2(q_2) + \frac{1-q_2}{2} \log_2(1-q_2)$

Given $: D = \frac{q_1+q_2}{2} \leq D^*$

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Rate Distortion / Binary Example –(2)

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* After (tough) manipulations, we find:
 $q_1 = q_2 = D^*$

* **Rate distortion lower bound:**
 $R(D^*) = 1 + D^* \log_2(D^*) + (1 - D^*) \log_2(1 - D^*)$

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Lossy Source Coding Theorem

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* **Interpretation: No compression strategy (for this memoryless source) performs better than the rate-distortion bound**

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Gaussian Continuous Sources

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* **For the memoryless Gaussian continuous source, we can derive the often used result:**

$$R(D) = \begin{cases} \frac{1}{2} \log_2 \left(\frac{\sigma_X^2}{D} \right) & 0 < D \leq \sigma_X^2 \\ 0 & D > \sigma_X^2 \end{cases}$$

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Interpretation of $R(D)$ Function

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1 At rate R_1 , the minimal distortion to be expected is D_1
2 To achieve distortion D_2 , the minimal rate R_2 is required

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R(D) Function as SNR Function

$$\left\{ \begin{aligned} R &= \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{D} \right) = \frac{1}{2} \log_2 \left(\frac{\sigma_x^2}{\sigma_q^2} \right) \\ \text{SNR} &= 10 \log_{10} \left(\frac{\sigma_x^2}{\sigma_q^2} \right) \end{aligned} \right. \Rightarrow \text{SNR} = 6.02R$$

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Remarks on Rate Distortion R(D)

- * Rate-distortion bound is **most relevant for continuous sources** as these can never be reconstructed perfectly
- * Although multimedia data is digitized (~large number of discrete amplitudes), theory of **continuous** sources is virtually always used
- * Rate-distortion theory does **not tell us how** to realize a system that achieves the bound (~best possible compression)
- * There are **few cases** for which R(D) can be calculated
- * R(D) is a convex function $0 \leq R(D) \leq H(X)$

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5LSE0 Mod 04, Part 4 How Well Do Scalar Quantizers Approach the Rate-Distortion Bound?

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Comparison w. Designed Quantizers – (1)

$$R = \frac{1}{2} \log_2 \left(\frac{1}{\sigma_q^2} \right)$$

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Comparison w. designed Quantizers – (2)

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Can We Explain the “Gap”?

```

    graph LR
      X["x(n)  
Amplitude continuous"] --> SQ["Scalar Quantizer"]
      SQ --> Y["y_j(n)  
Amplitude discrete"]
      Y --> EC["Entropy Coding"]
      EC --> Out["00101101"]
  
```

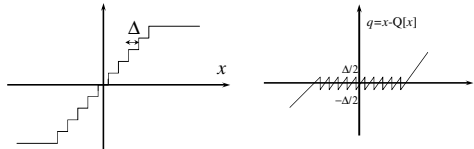
- * The above scheme is a **practical realization** of a compression system
- * **What about the rate-distortion behavior?**
 - What is the entropy of the quantizer outputs?
 - How is this related to the behavior of the quantizer?

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Approximate Quantizer Distortion – (1)

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* Approximate behavior of a uniform quantizer (at high bit rates): $p_q(q) \sim$ uniform pdf



$$p(q) = p(x - Q[x]) \approx \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases}$$

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Approximate Quantizer Distortion – (2)


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* For Lloyd-Max and Uniform Threshold Quantizers this approximation also holds “reasonably well”

* Distortion D for given Δ is now approximately equal to:

$$D = \sigma_q^2 = \frac{1}{12} \Delta^2$$

R	$\Delta (\sigma_x=1)$			SNR (dB)=10log ₁₀ (1/σ _q ²)		
	Uniform	Gaussian	Laplace	Uniform	Gaussian	Laplace
4	0.217	0.335	0.461	24.08 (24.06)	19.38 (20.29)	15.96 (17.51)
5	0.108	0.188	0.280	30.10 (30.12)	24.57 (25.30)	20.60 (21.84)
6	0.054	0.104	0.166	36.12 (36.14)	29.83 (30.45)	25.36 (26.38)
7	0.027	0.057	0.096	42.14 (42.16)	35.13 (35.67)	30.23 (31.14)
8	0.013	0.031	0.055	48.17 (48.51)	40.34 (40.96)	35.14 (35.98)

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Entropy of Quantizer Output Levels


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* For the quantizer representation levels, it holds

$$P(Y = y_j) = \int_{x_j}^{x_{j+1}} p(x) dx \approx \Delta p(x = y_j)$$

* This leads to the following approximation for the entropy of the quantizer outputs:

$$H(Y) = -\sum_{j=1}^N P[Y = y_j] \log_2 P[Y = y_j] \approx H(X) - \log_2(\Delta)$$

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Entropy of Quantizer Output Levels (2)


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* Entropy of quantizer outputs, i.e. the representation levels or quantized values):

$$H(Y) = H(X) - \log_2(\Delta)$$

(Resulting) entropy of quantized signal (Given) entropy of signal to be coded Influence of quantizer accuracy

- **More accurate** quantization generates symbols with **more entropy**.
- Similar results can be obtained for other quantizer types with sufficiently smooth PDFs and at high bit rates

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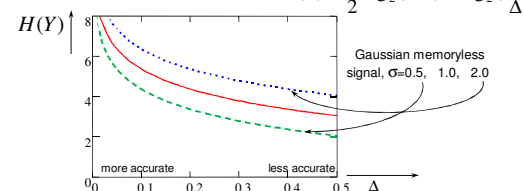
Application to Gaussian Source


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* We have : $H(Y) = H(X) - \log_2(\Delta)$

and for Gaussian sources : $H(X) = \frac{1}{2} \log_2(2\pi e) + \log_2(\sigma_x)$

yielding : $H(Y) = \frac{1}{2} \log_2(2\pi e) + \log_2\left(\frac{\sigma_x}{\Delta}\right)$



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Compare to Rate-Distortion Function – (1)

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* For a Gaussian memoryless signal, the quantizer with step size Δ generates:

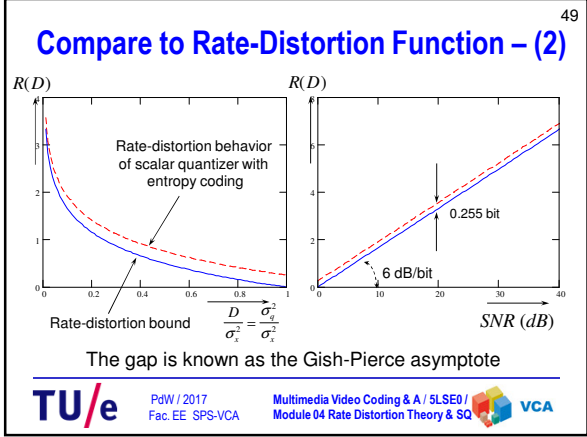
- Symbols with entropy: $H(Y) = \frac{1}{2} \log_2(2\pi e) + \log_2\left(\frac{\sigma_x}{\Delta}\right)$
- Distortion: $D = \sigma_q^2 = \frac{1}{12} \Delta^2$

* Combining these two gives:

$$H(Y) \approx \frac{1}{2} \log_2\left(\frac{\sigma_x^2}{D}\right) + \frac{1}{2} \log_2\left(\frac{e\pi}{6}\right)$$

= $R(D)$ + $\frac{1}{2} \log_2\left(\frac{e\pi}{6}\right)$ **the Gap**

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Exceed Gish-Pierce Asymptote?

- * Quantize multiple amplitudes *at the same time*
Vector Quantization (VQ)
- * The rate-distortion bound can be approximated arbitrarily closely by taking larger vector lengths N
- * VQ is a special topic, rarely used in practice

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