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Introduction Medical Image Processing (5XSA0), Module 02

Discrete Fourier Transform and Filtering

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slides version 1.0

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Overview Module 02


- * **Image Frequency Domain**
 - Fourier transform
 - Properties
- * **Image filtering based on frequency domain**
 - Various filter types and their response (Gaussian, Laplace..)
 - Noise reduction by frequency domain filtering

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Module 02 – Part 1 Frequency domain representation

Sampling and Fourier transform, 2-D extensions, convolution, aliasing, filtering concept

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Image waveform concept of Fourier

Image: essentially a 2-D signal waveform

- * **Signal:** decomposed in a sum of components
 - Fourier found in 1807 that any periodic or 'limited energy' waveform: can be a weighted sum of sines and cosines

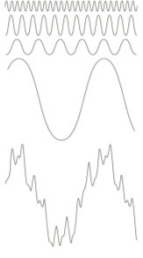



FIGURE 4.3 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

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Fourier series and impulses

Characterization of a video signal $f(t)$


- * Assume function $f(t)$ with period T , which can be expressed as **sum of sines and cosines**, with **coefficients** ($n=0, +/- 1, +/- 2, \dots$)

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n t / T} \quad c_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{-j2\pi n t / T} dt$$
- * **2. Unit impulses**

$$\delta(t) = 0 \quad (t \neq 0); \quad \delta(0) = \infty; \quad \int_{-\infty}^{+\infty} \delta(t) dt = 1$$
- * **3. Sifting process**

$$\int_{-\infty}^{+\infty} f(t) \delta(t - t_0) dt = f(t_0)$$

* Exercise: derive discrete versions!

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Signal is series of (weighted) impulses

Characterization of a video signal

- * Assume signal $s(t)$ with sampling period T , which can be expressed as **sum impulses**, with **signal coefficients** ($n=0, +/- 1, +/- 2, \dots$)

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{+\infty} s_n \delta(t - n\Delta T)$$

* If weights are $s_n=1$, then equal to sampling function!

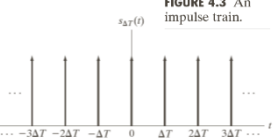



FIGURE 4.3 An impulse train.

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Fourier transform (one cont. variable)

Fourier transform of a signal $f(t)$

- * Assume function $f(t)$ with time t , which

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\omega t} dt$$

- * 2. **Inverse Fourier transform**

$$f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{+j2\pi\omega t} d\omega$$

- * 3. The equations form a transform pair. Note that with Euler, it also holds that $e^{j2\pi\omega t} = \cos(2\pi\omega t) + j \sin(2\pi\omega t)$

Note that ω is here frequency f in many books!

If $f(t)$ is real, then the transform is **complex!**

Fourier transform (cont.) - Example

$$F(\omega) = \int_{-W/2}^{+W/2} A e^{-j2\pi\omega t} dt = \frac{-A}{j2\pi\omega} [e^{-j2\pi\omega t}]_{-W/2}^{+W/2} = AW \frac{\sin(\pi\omega W)}{(\pi\omega W)}$$

Frequency spectrum is magnitude of transform!

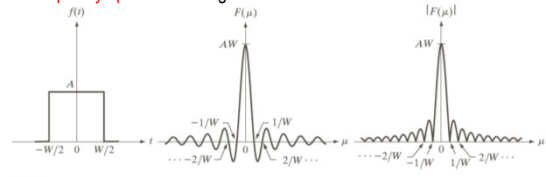


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

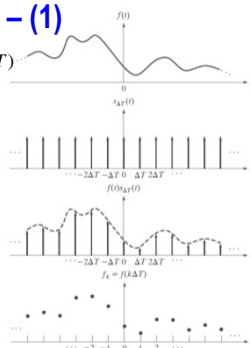
Sampling of a signal - (1)

$$f_n = f(n\Delta T) = \sum_{k=-\infty}^{+\infty} f(t)\delta(t - n\Delta T)$$

Sampling of

- continuous signal $f(t)$
- time period ΔT
- results in weighted impulse train

FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)



Sampling of a signal - (2)

Sampling speed such that spectra fit!

- Signal should be **bandwidth-limited**

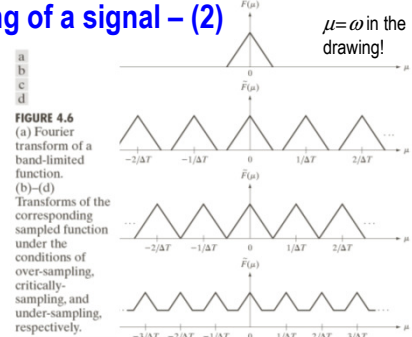


FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)-(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

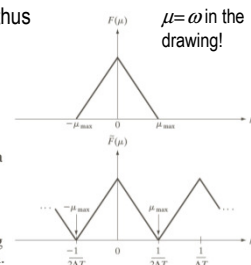
Sampling of a signal - (3) Theorem

Reconstruction is possible if one spectrum can be extracted, thus

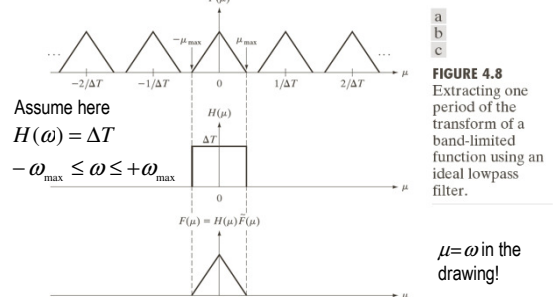
$$2 \frac{1}{2\Delta T} > 2\omega_{\max}$$

Sampling at $2x$ ω_{\max} is called Nyquist rate

FIGURE 4.7 (a) Transform of a band-limited function. (b) Transform resulting from critically sampling the same function.



Sampling Theorem (& Nyquist rate)



Assume here $H(\omega) = \Delta T$

$$-\omega_{\max} \leq \omega \leq +\omega_{\max}$$

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

$\mu = \omega$ in the drawing!

Under-Sampling

Under-sampling gives **overlapping** frequency areas

- HF becomes LF!
- Overlap is called **aliasing**
- Aliasing always occurs but can be suppressed with **low-pass pre-filters**

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

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Module 02 – Part 2 Sampling and aliasing

Sampling frequency aspects using the cosine waveform as a basis

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Sampling and aliasing – (2)

$\cos(2\pi \cdot 200t)$ and $\cos(2\pi \cdot 1200t)$ sampled @ $F_s = 500$ samples/sec

Note that the phase changes the sampling values!

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Concept of aliasing – (1)

Other frequencies resulting in same samples (e.g. as in right figure)?
Use property $\cos(\theta \cdot n) = \cos((\theta + 2\pi) \cdot n) \Rightarrow$ **Aliasing** 200 & 1200 together $f_s = 500$

$$x_1[n] = \cos(2\pi \cdot 200 \cdot n \cdot \frac{1}{1000}) = \cos(0.4\pi \cdot n) = \cos((0.4\pi + 2\pi) \cdot n) = \cos(2.4\pi \cdot n) = \cos(2\pi \cdot 1200 \cdot n \cdot \frac{1}{1000})$$

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Spectrum of discrete-time signal – (1)

Spectrum of $x_1[n] = \cos(0.4\pi n) = (\frac{e^{j0.4\pi n} + e^{-j0.4\pi n}}{2})$

Spectrum analog cosine two components, here ∞ components

One interval length 2π relevant: **Fundamental Interval (FI)** $\{-\pi, \pi\}$

Example: Two different discrete-time signals, same spectra

Spectrum $y_1[n] = 2\cos(0.4\pi) + \cos(0.6\pi)$ Spectrum $y_2[n] = 2\cos(0.4\pi) + \cos(2.6\pi)$

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Spectrum of discrete-time signal – (3)

Example: $x(t) = \cos(2\pi(600)t)$ ($f_{max} = 600$), different sample rates

Oversampling case: $f_s = 2000$ samples/sec ($> 2 \cdot f_{max}$)

Undersampling case: $f_s = 500$ samples/sec ($< 2 \cdot f_{max}$)

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Reconstruction – (1)

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Relation between analog frequency (ω) and discrete-time frequency (θ):
 $\omega = 2\pi f$ while $\theta = 2\pi(f/f_s)$ thus $\omega = \theta \cdot f_s$

⇒ Each discrete-time frequency mapped to continuous-time frequency

Take care alias terms → Select (e.g.) FI = $\{-\pi, \pi\} \approx \{-\frac{f_s}{2}, \frac{f_s}{2}\}$

Chirp

How does $x(t)$ and $\hat{x}(t)$ sound when $f_s = 2000$ samples/sec?

Explanation:

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Ideal reconstruction (Shannon's theorem)

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Conclusion from Shannon Sampling Theorem

If $x(t)$ contains no frequencies higher than f_{max} and $f_s > 2 \cdot f_{max}$ output $\hat{x}(t)$ of ideal D-to-C converter equal to input $x(t)$ ideal C-to-D converter

Why sample rate audio CD 44.1 kHz?
 → 20 kHz upper limit human hearing/ perception musical signals

What happens if we don't sample fast enough? → Aliasing

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Discrete-to-Continuous conversion – (2)

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How interpolate smooth continuous-time function through samples $x[n]$?

How reconstruct from discrete-time to analog domain?

For sinusoidal signals (mathematical expression of samples $\equiv \cos(\dots)$)

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Different pulses for non-ideal D-to-C conv.

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Square: $p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$

Triangular: $p(t) = \begin{cases} 1 - \frac{|t|}{T_s} & -T_s < t \leq T_s \\ 0 & \text{otherwise} \end{cases}$

Cubic: Third order polynomial with $p(t)$ for $t = \pm T_s, \pm 2T_s$ and derivative smooth at sample locations

How make smooth reconstruction?
 Use smooth pulse with long duration or use oversampling → original waveform does not vary much over duration of $p(t)$.

Note practical consequences of all different choices

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Discrete-to-Continuous Conversion rev.

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How does D-to-C work in general?

Adding, possible overlapping, shifted pulses
 $\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n]p(t - nT_s)$ with $p(t)$ characteristic pulse shape converter. Thus each $t_n = n \cdot T_s$ a pulse $p(t - nT_s)$ is emitted with amplitude $x[n]$

Four different pulses for D-to-C with $T_s = 0.005$ [sec] (i.e. $f_s = 200$ [Hz])

Pulse	Duration
Square	T_s
Triangle	$2T_s$
Cubic	$4T_s$
Ideal	∞

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Summary Sampling and Aliasing

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- Sampling sinusoid: With $x(t) = A \cos(\omega t + \phi)$ and $T_s = 1/f_s$
 $x[n \cdot T_s] = x(t)|_{t=n \cdot T_s} = A \cos((\omega \cdot T_s) \cdot n + \phi) = A \cos(\theta \cdot n + \phi)$
 - Absolute frequency: $\omega = 2\pi f$ [rad/sec] (f [Hz])
 - Relative frequency: $\theta = \omega \cdot T_s = 2\pi(f/f_s)$ [rad] (dimensionless)
- Aliasing frequencies $\cos(\theta_0 n + \phi)$ with relative frequency θ_0 :
 ⇒ $(\theta_0 + 2\pi) n + \phi$ or $(2\pi l - \theta_0) n - \phi$ with integer l
- Spectrum discrete-time signal: **Fundamental Interval (FI)**
 FI = $\{-\pi, \pi\} \approx \{-\frac{f_s}{2}, \frac{f_s}{2}\}$, periodic with ∞ components
- Ideal reconstruction sinusoidal signal: Each discrete-time frequency θ mapped via $\omega = \theta \cdot f_s$ to continuous-time frequency. Take care of alias term by selecting one period ⇒ **Choose one FI**
- Sampling theorem:
 Continuous-time signal $x(t)$ with frequencies **no higher than f_{max}** can be reconstructed exactly from its samples $x[n] = x(t)|_{t=n \cdot T_s}$, if samples are taken at a rate $f_s = 1/T_s$, that is **greater than $2f_{max}$**

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Module 02 – Part 3 Definition of Fourier Transform

Discrete Fourier transform, 2-D extensions, image and frequency domain, aliasing, DFT filtering concepts

Discrete Fourier Transform (one variable)

Discrete Fourier Transform of a signal $f(t)$

* Assume function $f(t)$ sampled at time ΔT

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \quad m = 0, 1, 2, \dots, M-1$$

* 2. **Inverse Discrete Fourier transform**

$$f_n = \sum_{m=0}^{M-1} F_m e^{+j2\pi mn/M} \quad n = 0, 1, 2, \dots, M-1$$

* 3. This can be derived with taking M samples in interval of frequency $0 \leq \omega \leq \frac{1}{\Delta T}; m = 0, 1, 2, \dots, M-1 \quad \omega = \frac{m}{M\Delta T}$

1-D Discrete Fourier Transform (mod. Not.)

Discrete Fourier Transform of an image signal $I(x, y)$

* Assume function $I(t)$ sampled at times $x\Delta T$ and frequency u

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0, 1, 2, \dots, M-1$$

* 2. **Inverse Discrete Fourier transform**

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{+j2\pi ux/M} \quad x = 0, 1, 2, \dots, M-1$$

* 3. This is the common notation in many books and remainder of the course!

Properties of 1-D DFT

* 1. Both DFT and IDFT are **infinitely periodic**

$$F(u) = F(u + kM) \quad f(x) = f(x + kM)$$

* 2. Assuming that $f(x)$ is real and m samples are taken, then the DFT gives a set of m complex numbers

* 3. It can be shown that the discrete equivalent of **convolution** is

$$f(x) \otimes h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

* 4. If $f(x)$ consists of M samples of $f(t)$ taken ΔT units apart, then period duration $T = M\Delta T$ and $\Delta u = 1/(M\Delta T) = 1/T$

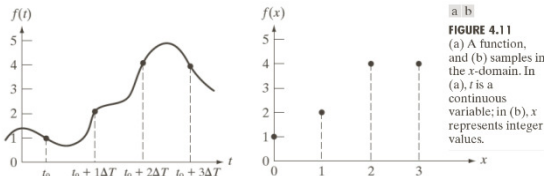
1-D DFT computation example

* 1. Compute $F(0)$, $F(1)$ etc. of the DFT

- $F(0) = 1+2+4+4 = 11$, $F(1) = -3+2j$,
- $F(2) = -(1+0j)$, $F(3) = -(3+2j)$

* 2. The inverse DFT gives the computation of $f(t)$

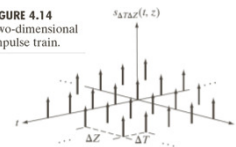
$$-f(0) = 1/4 [11 - 3 + 2j - 1 - 3 - 2j] = 1$$



Extensions to 2-D DFT – (1)

- * 1. Sampling over 2-D area (image) instead of signal and impulse train becomes 2-D signal
- * 2. DFT: summing over pixels (x, y) , and 2 param's m, n
- * DFT periodic in 2-D frequency domain, param's (u, v)
- * 2-D sampling theorem: 2 constraints for band-limitations

FIGURE 4.14 Two-dimensional impulse train.



$$\frac{1}{\Delta T} > 2\omega_{\max}; \quad \frac{1}{\Delta Z} > 2v_{\max}$$

Extensions to 2-D DFT – (2)

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- * 1. Ideal footprint of 2-D LPF becomes rectangular
- * 2. Aliasing becomes 2-D pattern (e.g. Moire)

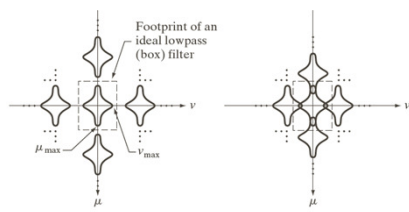


FIGURE 4.15
Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

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Extensions to 2-D DFT – (3) / Moire

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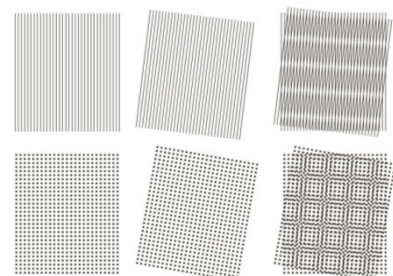


FIGURE 4.20
Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.

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Extensions to 2-D DFT – (4) / Moire

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FIGURE 4.21
A newspaper image of size 246 x 168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the ±45° orientation of the halftone dots and the north-south orientation of the sampling grid used to digitize the image.

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2-D Discrete Fourier Transform

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- * **2-D Discrete Fourier Transform** of image signal $f(x, y)$ for $0 \leq u \leq M-1$ and $0 \leq v \leq N-1$ is

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ax/M + by/N)}$$
- * **2-D Inverse Discrete Fourier Transform**

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{+j2\pi(ax/M + by/N)}$$
- * 3. The above equations hold for an image of size $M \times N$ pixels and together they form a transform pair

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Properties of 2-D DFT – (1)

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- * **2-D Spatial and frequency intervals** using image signal $f(x, y)$ of size $M \times N$

$$\Delta u = \frac{1}{M\Delta T}; \quad \Delta v = \frac{1}{N\Delta Z}$$
- * **Translation and rotation**

$$f(x, y) e^{+j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(u_0x/M + v_0y/N)}$$
- * Likewise, a rotation over an angle in $f(x, y)$ gives an equal rotation of $F(u, v)$ (this can be proven with polar coordinates with $x = r \cos(\theta)$ etc., and $u = w \cos(\phi)$, etc.)

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Properties of 2-D DFT – (2)

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- * **2-D Periodicity of the DFT spectra**

$$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = \text{etc.}$$

$$f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = \text{etc}$$
- * As a special case (use Euler rule for this with $u_0 = M/2$)

$$f(x, y) (-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$
- * As a result, the spectrum shifts such that the $F(0,0)$ is now at the origin!

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Properties of 2-D DFT – (3) Centering

Multiplication with $(-1)^{x+y}$ locates $F(0,0)$ in the origin

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u, v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A shifted DFT obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period, as in (b).

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Properties of 2-D DFT – (4) Symmetry

Spatial Domain [†]	Frequency Domain [†]
1) $f(x, y)$ real	$F^*(u, v) = F(-u, -v)$
2) $f(x, y)$ imaginary	$F^*(-u, -v) = -F(u, v)$
3) $f(x, y)$ real and even	$R(u, v)$ even; $I(u, v)$ odd
4) $f(x, y)$ real and odd	$R(u, v)$ odd; $I(u, v)$ even
5) $f(x, y)$ imaginary and even	$F^*(u, v)$ complex
6) $f(x, y)$ imaginary and odd	$F^*(-u, -v)$ complex
7) $f(x, y)$ real and even	$F(u, v)$ real and even
8) $f(x, y)$ real and odd	$F(u, v)$ imaginary and odd
9) $f(x, y)$ imaginary and even	$F(u, v)$ imaginary and even
10) $f(x, y)$ imaginary and odd	$F(u, v)$ real and odd
11) $f(x, y)$ complex and even	$F(u, v)$ complex and even
12) $f(x, y)$ complex and odd	$F(u, v)$ complex and odd

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

[†]Recall that x, y, u, v are discrete (integer) variables, with x and u in the range $[0, M-1]$, and y, v in the range $[0, N-1]$. To say that a complex function is *even* means that its real and imaginary parts are even, and similarly for an odd complex function.

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Fourier spectrum and phase 2-D DFT

- 2-D DFT is complex, it can be expressed in polar form

$$F(u, v) = |F(u, v)|e^{j\phi(u, v)}$$
- Where the magnitude is the **Fourier spectrum** and ϕ the **phase angle**

$$|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}$$

$$\phi(u, v) = \arctan\left[\frac{I(u, v)}{R(u, v)}\right]$$
- Finally, the **power spectrum** is

$$P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)$$

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Fourier spectrum and phase – Examples

- Typically, $f(x, y)$ is real, which implies that 2-D DFT spectrum is **even symmetric** about the origin

$$|F(u, v)| = |F(-u, -v)|$$
- and ϕ the **phase angle is odd symmetric** about the origin

$$\phi(u, v) = \phi(-u, -v)$$
- Finally, note that

$$|F(0, 0)| = \left| MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \right| = MN |\bar{f}(x, y)|$$

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Fourier spectrum – Example 1

FIGURE 4.24 (a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

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Fourier spectrum – Example 2a

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

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Fourier spectrum – Example 2b Phase

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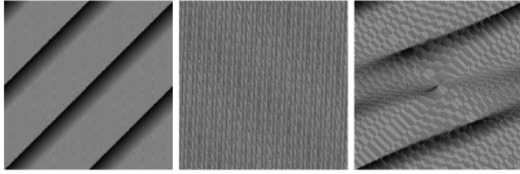


FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

Summary 2-D DFT Proper's 1

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Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = F(u, v) e^{j\phi(u, v)}$
4) Spectrum	$ F(u, v) = [R^2(u, v) + I^2(u, v)]^{1/2}$ $R = \text{Real}(F); I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1} \left[\frac{I(u, v)}{R(u, v)} \right]$
6) Power spectrum	$P(u, v) = F(u, v) ^2$
7) Average value	$\bar{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

TABLE 4.2 Summary of DFT definitions and corresponding expressions.

(Continued)

Summary 2-D DFT Proper's 2

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Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N)$ $= F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N)$ $= f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.

TABLE 4.2 (Continued)

Summary 2-D DFT Properties – (3)

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Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y) e^{j2\pi(ux_0/M + vy_0/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, ($M/2, N/2$)	$f(x, y) (-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v) (-1)^{x+y}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r \cos \theta \quad y = r \sin \theta \quad u = \omega \cos \varphi \quad v = \omega \sin \varphi$
6) Convolution theorem ¹	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v) H(u, v)$ $f(x, y) h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

TABLE 4.3 Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the closed-form, continuous expressions.

(Continued)

Summary 2-D DFT Properties – (4)

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- * And remember that due to Euler's formula and the cosine/sine rules:

$$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{j}{2} [\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)]$$

- * And

$$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2} [\delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0)]$$

- * Finally, note that

$$\delta(x, y) \Leftrightarrow 1$$

Module 02 – Part 4 Image Filtering based on DFT

Filtering concept using DFT, filter types, special cases

Frequency-domain filtering fundament

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- * Filtering techniques in the frequency domain are based on modifying the Fourier transform
- * Note that each value of $F(u, v)$ contains information of all image samples!
- * If a filter has a **filter transfer function**, we exploit that

$$g(x, y) = IDFT[H(u, v)F(u, v)]$$

Where F , H and g are all matrices of $M \times N$. Assume that H and F are centered, requiring pre-processing!

Necessity of padding with zeros – (1)

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- * Without any measures, the use of the DFT creates a wraparound error when periodicity is created
- * The solution is to enlarge the the window with extra samples, such as **padding with zeros** (creates a border)
- * **Padding can be used to decrease ringing** in the frequency domain, employing the extended filter impulse response in the time domain.

Necessity of padding zeros – (2)

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Example

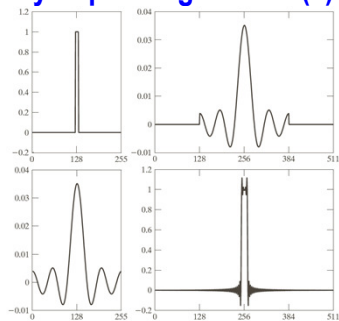


FIGURE 4.34 (a) Original filter specified in the (centered) frequency domain. (b) Spatial representation obtained by computing the IDFT of (a). (c) Result of padding (b) to twice its length (note the discontinuities). (d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). (The curves appear continuous because the points were joined to simplify visual analysis.)

Summary of steps for DFT-based filters

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1. Determine padding parameters, typically $P=2M$ and $Q=2N$
2. Form padded image $f_p(x, y)$ of size $P \times Q$ appending zeros
3. Multiply padded image with $(-1)^{x+y}$ for centering spectrum
4. Compute the DFT, $F(u, v)$, of the centered padded image
5. Generate a real, symmetric filter, $H(u, v)$, of size $P \times Q$ and center in the middle, and compute $G(u, v) = H(u, v)F(u, v)$
6. Obtain the processed image with the IDFT of $G(u, v)$ and take the real part, and re-center it

$$g_p(x, y) = \{real[IDFT[G(u, v)]]\}(-1)^{x+y}$$
7. Extracting the $g(x, y)$ result by selecting the $M \times N$ region from the top-left quadrant

Example of DFT-based filters / Gaussian

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$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2} \Leftrightarrow H(u) = A e^{-u^2/2\sigma^2}$$

Both real functions!

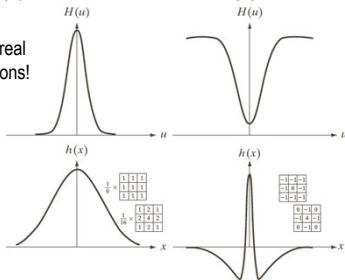


FIGURE 4.37 (a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

Example of DFT-based filters / Gaussian

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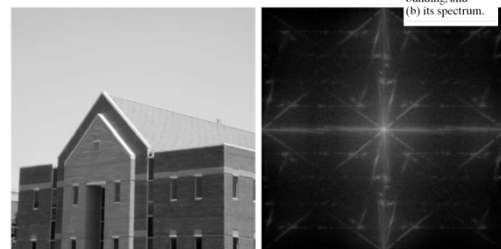


FIGURE 4.38 (a) Image of a building, and (b) its spectrum.

Sobel filter as DFT-based filter example

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FIGURE 4.39 (a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

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Ideal 2-D LPF filter – (1) Concept

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- Passes all frequencies within circle of radius D_0 : $H(u,v)=1$
- Rejects all frequencies outside circle with radius D_0 : $H(u,v)=0$

$$D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2 \right]^{1/2}$$

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

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Ideal 2-D LPF filter –(2)

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Performance comparison shows

- this is not very practical, as
- most detail is removed in only 13% of the power
- and ringing occurs at already 2% removal

FIGURE 4.42 (a) Original image. (b-f) Results of filtering using ILPFs with cutoff frequencies at radii values 10, 20, 40, 80, and 160, as shown in Fig. 4.41(b). The power removed by these filters was 13.6%, 4.3, 2.2, and 0.6%, of the total, respectively.

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Ideal 2-D LPF filter – (3) Explanation

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1. Blur comes from the main lobe in impulse response
2. Ringing comes from the side lobes in impulse response
3. Spread of sinc function is inversely proportional to pass-band of $H(u,v)$. (Discuss the trade-off)

FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000×1000 . (b) Intensity profile of a horizontal line passing through the center of the image.

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Smooth 2-D LPF filter – (1) Gaussian

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1. Establish smooth roll-off in transfer function $H(u,v)$
2. Ringing comes from the side lobes in impulse response

$$H(u,v) = e^{-D^2(u,v)/2D_0^2} \quad (\text{note: } \sigma = D_0)$$

FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0 .

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Gaussian 2-D LPF filter – (2)

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1. Blur can be regulated by variance
2. Ringing is absent
3. Spread of sinc function is smooth
4. Good visual results

FIGURE 4.48 (a) Original image. (b-f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

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2-D HPF filters – various forms

$H_{HP}(u, v) = 1 - H_{LP}(u, v)$
 $H_{HP}(u, v) = 1 \quad : D(u, v) > D_0$
 $H_{HP}(u, v) = 0 \quad : D(u, v) \leq D_0$

HPF is opposite of LPF filter!

FIGURE 4.52 Top row: Perspective plot, image representation and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

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Smooth 2-D HPF filter – cf. Gaussian

$H_{HP}(u, v) = 1 - e^{-D^2(u, v) / 2D_0^2}$ (note: $\sigma = D_0$)

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

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2-D HPF filters – perform.

Ideal HPF vs. Gaussian HPF performance

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60,$ and 160 .

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60,$ and 160 , corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

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Laplacian filtering - Concept

- Laplacian can be implemented in frequency domain with

$$H(u, v) = -4\pi^2(u^2 + v^2)$$
- Or, with respect to center of frequency rectangle

$$H(u, v) = -4\pi^2[(u - P/2)^2 + (v - Q/2)^2] = -4\pi^2 D^2(u, v)$$
- Furthermore, it holds that for the filter in the freq. domain

$$\nabla^2 f(x, y) = IDFT\{H(u, v)F(u, v)\}$$
- Enhancement with subjective sharpness is obtained when adding this term (note the scaling), so that

$$g(x, y) = f(x, y) + c\nabla^2 f(x, y)$$

$$g(x, y) = IDFT\{F(u, v) - H(u, v)F(u, v)\} = IDFT\{[1 - H(u, v)]F(u, v)\} = IDFT\{[1 + 4\pi^2 D^2(u, v)]F(u, v)\}$$

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Laplacian filtering - performance

FIGURE 4.58 (a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

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Unsharp masking & highboost filtering

- Concept: sharpen images by **subtracting an unsharp image** from the original. The difference is **the mask**
- Steps are:
 - Blur the original image
 - Subtract the blurred image from original (difference is the mask)
 - Add mask to the original
- More formally, the specification is

$$g_{mask}(x, y) = f(x, y) - \bar{f}(x, y)$$

$$g(x, y) = f(x, y) + k * g_{mask}(x, y)$$
- If $k=1$, it is unsharp masking, if $k>1$, we have highboost filtering, with if $k<1$, we have high-freq. emphasis

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High-frequency emphasis filtering

- For the unsharp mask, we can take the freq.-domain approach

$$f_{LP}(x, y) = IDFT[H_{LP}(u, v)F(u, v)]$$
- Function $f_{LP}(x, y)$ is a low-pass smoothed image from the filtering with $H_{LP}(u, v)$.
- The unsharp masking equation from the previous slide becomes now **in the frequency domain**

$$g(x, y) = IDFT\{[1 + k * [1 - H_{LP}(u, v)]]F(u, v)\}$$
- This result can be respecified as a **high-pass filter result**

$$g(x, y) = IDFT\{[1 + k * H_{HP}(u, v)]F(u, v)\}$$

$\xleftrightarrow{\text{High-frequency emphasis filter}}$

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High-freq. emphasis filtering / Example

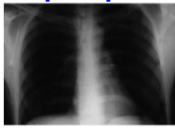
- For more general form, take coefficients k_1, k_2 such that

$$g(x, y) = IDFT\{[k_1 + k_2 * H_{HP}(u, v)]F(u, v)\}$$
- Where $k_1 > 0$ gives controls of the offset of the origin and $k_2 > 0$ the high-frequency contribution
- In medical applications, ringing artifacts from filters are not accepted. Because spatial and freq.-domain Gaussian filters are transform pairs, these filters give smooth response and avoid ringing
- In the following example, HF emphasis filtering helps


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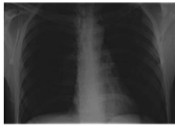
High-freq. emphasis filtering / Example




HF emphasis filter, $k_1=0.5$, $k_2=0.75$



Gaussian filter with $D_0=40$





HF emphasis & histogram equaliz.

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

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Module 02 – Appendix

Examples of simple transformations

Affine transform, Basic principles of orthogonal transforms, Matrix notation

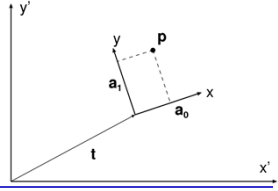
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Affine 2D Coordinate Transform – (1)

- * Express coordinate frame (x, y) in coordinate frame (x', y') .
- * In motion estimation:
 - (x, y) could be the coordinate in the last frame, and
 - (x', y') the corresponding coordinate in the current frame.

$$\begin{aligned}
 \mathbf{p} &= (x \ y)^\top \\
 \mathbf{p}' &= (x' \ y')^\top \\
 \mathbf{p}' &= \mathbf{t} + x \cdot \mathbf{a}_0 + y \cdot \mathbf{a}_1 \\
 &= (\mathbf{a}_0 \ \mathbf{a}_1) \mathbf{p} + \mathbf{t} \\
 &= \mathbf{A} \mathbf{p} + \mathbf{t}
 \end{aligned}$$



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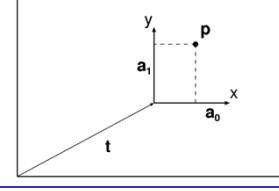
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Affine 2D Coordinate Transform – (2)

- * Special (simple) case 1: translation only

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \longrightarrow \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
- * \mathbf{A} is identity matrix \mathbf{I}

$$\begin{aligned}
 \mathbf{p}' &= \mathbf{A} \mathbf{p} + \mathbf{t} \\
 &= \mathbf{p} + \mathbf{t}
 \end{aligned}$$



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Affine 2D Coordinate Transform – (3) 73

- Special case 2: scaling only**
 - scaling in x direction scaling in y direction

$$\mathbf{a}_0 = \begin{pmatrix} s_x \\ 0 \end{pmatrix} \quad \mathbf{a}_1 = \begin{pmatrix} 0 \\ s_y \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

- Isotropic scaling (rigid motion)**

$$s_x = s_y = s \quad \mathbf{A} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \quad \mathbf{p}' = \mathbf{A}\mathbf{p}$$

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Affine 2D Coordinate Transform – (4) 74

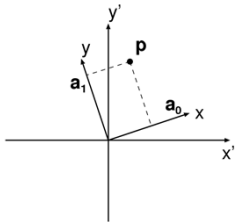
- Special case 3: rotation only**
 - constraint 1: size stays constant

$$\|\mathbf{a}_0\| = \|\mathbf{a}_1\| = 1$$

- constraint 2: coordinate axes are perpendicular

$$\mathbf{a}_0^\top \cdot \mathbf{a}_1 = 0$$

- A is an orthonormal matrix (rotation matrix R)**



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Affine 2D Coordinate Transform – (5) 75

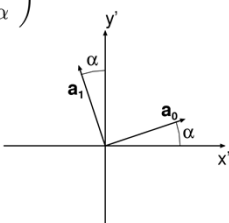
- Rotation matrix for rotation by an angle α**

$$\mathbf{a}_0 = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \quad \mathbf{a}_1 = \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

- rotation by $-\alpha$**

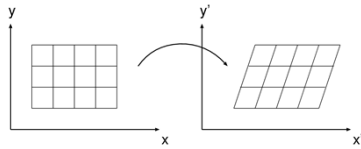
$$\mathbf{A}^{-1} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

$$\mathbf{A}^{-1} = \mathbf{A}^\top$$


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Affine 2D Coordinate Transform – (6) 76

- Image skew (horizontal)**

$$\mathbf{A} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

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Affine 2D Coordinate Transform – (7) 77

- Concatenation of transforms**

$$\mathbf{p}' = \mathbf{A}_1\mathbf{p} + \mathbf{t}_1$$

$$\mathbf{p}'' = \mathbf{A}_2\mathbf{p}' + \mathbf{t}_2$$

- gives** $\mathbf{p}'' = \underbrace{\mathbf{A}_2\mathbf{A}_1}_{\mathbf{A}_3}\mathbf{p} + \underbrace{\mathbf{A}_2\mathbf{t}_1 + \mathbf{t}_2}_{\mathbf{t}_3}$

(Shortly we will introduce a trick to make this much easier)

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Affine 2D Coordinate Transform – (8) 78

- Computing the inverse of**

$$\mathbf{p}' = \mathbf{A}\mathbf{p} + \mathbf{t}$$

- gives** $\mathbf{p} = \mathbf{A}^{-1}\mathbf{p}' - \mathbf{A}^{-1}\mathbf{t}$

(the mathematical trick will make this much easier, too)

- Inverse is possible if A is nonsingular.**

Set of nonsingular affine transforms constitutes a *group*.

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Affine 2D Coordinate Transform – (9)


* Here the trick: **augment each point-coordinate with a dummy '1'**:

$$\mathbf{p} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

* Now, write affine transform as simple matrix multiplication:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{00} & a_{01} & t_x \\ a_{10} & a_{11} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathbf{p}' = \mathbf{A}\mathbf{p} + \mathbf{t}$$

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Affine 2D Coordinate Transform – (10)


* Concatenation is now very easy

$$\mathbf{A}_3 = \mathbf{A}_2\mathbf{A}_1 \quad (\mathbf{A}_i \text{ are } 3 \times 3 \text{ matrices})$$

* And the inverse transform is simply the 3x3 matrix inverse.

$$\mathbf{p}' = \mathbf{A}\mathbf{p} \quad \mathbf{p} = \mathbf{A}^{-1}\mathbf{p}'$$

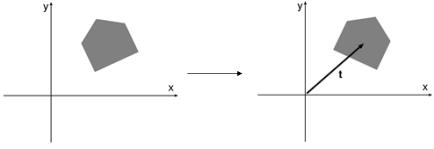
* This useful “trick” will reappear later...


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Affine 2D Coordinate Transform Example: Transform Sequence – (1)

* What is the transformation between these two images?

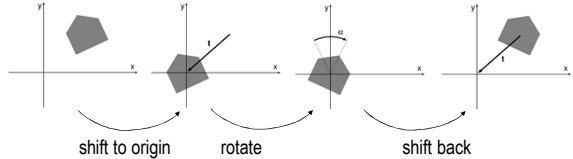


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Affine 2D Coordinate Transform Example: Transform Sequence – (2)

* Break the transformation into sequence of elementary steps.



$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

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Affine 2D Coordinate Transform Example: Transform Sequence – (3)

* Successive steps:

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

* Combined transform:

$$\mathbf{p}' = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1\mathbf{p}$$

* Multiplied together:

$$\begin{pmatrix} t_x \cos \alpha & -t_y \sin \alpha & -t_x \\ t_x \sin \alpha & t_y \cos \alpha & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

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