Enabling Technologies for Sports (5XSF0) Module 3

Frequency domain processing

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What do we consider in frequency domain processing?

- Filtering in the frequency domain via the Fourier transform
  - can be used for image enhancement, restoration, compression
- How to perform frequency domain processing in Matlab

(The slides are based on “Digital Image Processing Using Matlab”, R. C. Gonzalez, R. E. Woods, S. L. Eddins)

2D Discrete Fourier Transform

- For image ***f(x,y)*** (***x*=0,1,2,…,*M*-1** and ***y***=0,1,2,…,***N***-1), discrete Fourier transform (DFT) is

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

where u=0,1,2,…,M-1 and v=0,1,2,…,N-1

- Frequency domain is the coordinate system spanned by ***F(u,v)*** with ***u*** and ***v*** as frequency variables

Inverse Discrete Fourier Transform

- Inverse DFT is given by

$$f(x,\ y)\ =\ \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,\ v)e^{j2\pi(ux\ /\ M+vy\ /\ N)}$$

where x=0,1,2,…,M-1 and y=0,1,2,…,N-1,

the values $F(u,v)$ are called Fourier coefficients

- ***F(0,0)*** is the DC component (Direct current – from electrical engineering) of the Fourier transform

Analyzing a transform – (1)

- Even if $f(x,y)$ is real, the transform in general is complex
- Spectrum – the magnitude of $F(u,v)$ – is the principal method of visually analyzing a transform
- Fourier spectrum is defined as

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

where $R(u,v)$ and $I(u,v)$ represent the real and imaginary components of $F(u,v)$

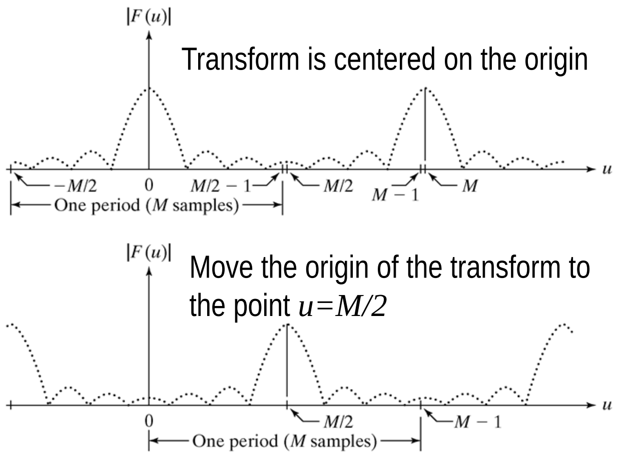
Analyzing a transform – (2)

- Fourier spectrum is symmetric about the origin

$$|F(u,v)| = |F(-u,-v)|$$

- DFT is infinitely periodic in both u and v directions, the periodicity is determined by M and N
- Image obtained by taking the inverse DFT is also infinitely periodic; DFT implementations compute only one period – $M \times N$

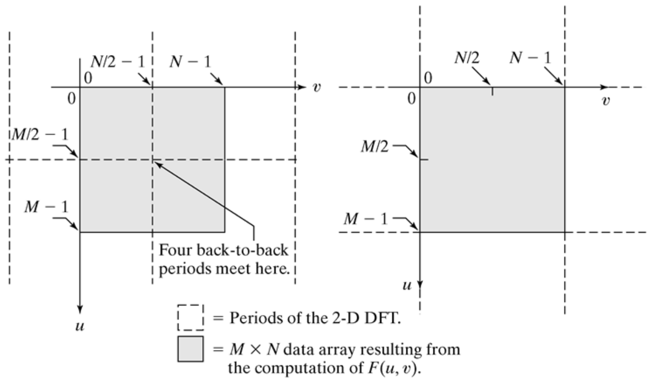
Analyzing a transform – (3)



a
b

FIGURE 4.1 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Centered spectrum in the same interval, obtained by multiplying $f(x)$ by $(-1)^x$ prior to computing the Fourier transform.

Analyzing a transform – (4)



$N/2-1$
$N-1$
0
0
v
$M/2-1$
$M-1$
Four back-to-back periods meet here.
u
$N/2$
$N-1$
0
0
v
$M/2$
$M-1$
u
= Periods of the 2-D DFT.
= $M \times N$ data array resulting from the computation of $F(u, v)$.

In Matlab, data is rearranged after the transform using `fftshift`

a b

FIGURE 4.2 (a) $M \times N$ Fourier spectrum (shaded), showing four back-to-back quarter periods contained in the spectrum data. (b) Spectrum obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ prior to computing the Fourier transform. Only one period is shown shaded because this is the data that would be obtained by an implementation of the equation for $F(u, v)$.

Computing and visualizing 2D DFT in Matlab

* DFT and its inverse are obtained in practice using a Fast Fourier Transform (FFT): **`F=fft2(f)`**
  - it is necessary to pad the input image with zeros: `F=fft2(f,P,Q)` – pads the input so that the resulting function is of size P x Q
  - Fourier spectrum: `S=abs(F)`
  - Inverse Fourier transform: `f=ifft2(F);`
    - obtain an image containing only real values: `f=real(ifft2(F))`

Computing and visualizing 2D DFT in Matlab: example

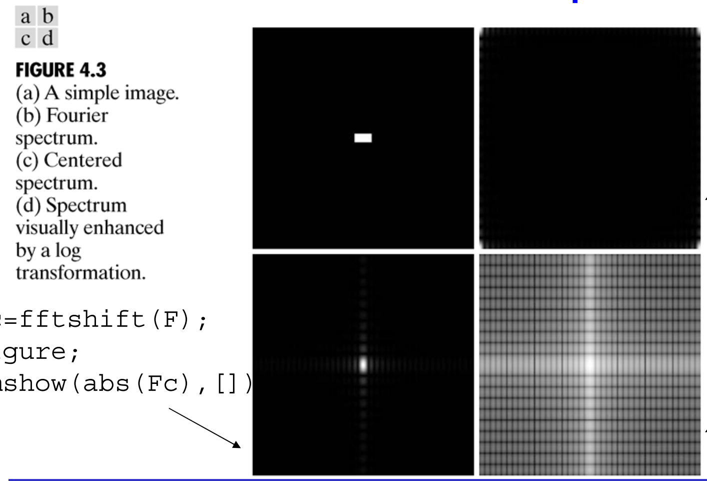


FIGURE 4.3
(a) A simple image.
(b) Fourier spectrum.
(c) Centered spectrum.
(d) Spectrum visually enhanced by a log transformation.

```
F=fft2(f);
S=abs(F);
figure;
imshow(S,[])
```

```
Fc=fftshift(F);
figure;
imshow(abs(Fc),[])
```

```
S2=log(1+
+abs(Fc));
figure;
imshow(S2,[])
```

Filtering in the frequency domain

- Convolution theorem

$$f(x,y)*h(h,y) \Leftrightarrow H(u,v)F(u,v)$$

and

$$f(x,y)h(h,y) \Leftrightarrow H(u,v)*F(u,v)$$

symbol "$*$" indicates convolution

H(u,v) – filter transfer function

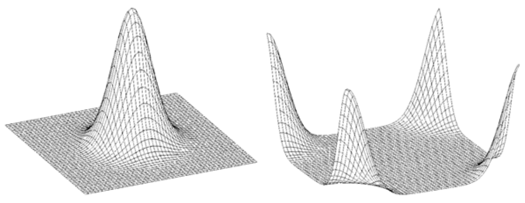
- Frequency domain filtering: select a filter transfer function that modifies ***F(u,v)*** in a specified manner

Transfer function for lowpass filter: example

a b

FIGURE 4.4 Transfer functions of (a) a centered lowpass filter, and (b) the format used for DFT filtering. Note that these are frequency domain filters.

after `fftshift`



Lowpass filter attenuates the high-frequency components of *F(u,v)*, while leaving the low frequencies relatively unchanged.

Result of lowpass filtering is image blurring (smoothing).

Padding functions with zeros

For functions $f(x,y)$ and $h(x,y)$ of size A x B and C x D respectively, form two extended (padded) functions, both of size P x Q, by appending zeros to f and g.

Wraparound error is avoided by choosing $P \geq A + C - 1$ and $Q \geq B + D - 1$

If the functions are of the same size, M x N, then the padding values are $P \geq 2M - 1$ and $Q \geq 2N - 1$

Basic steps (1-2) in DFT filtering

Step-by-step procedure involving Matlab functions, where
f – image to be filtered,
g – result,
$H(u,v)$ – filter function of the same size as the padded image.

1. Obtain the padding parameters: `PQ=2*size(f).`
2. Obtain the Fourier transform with padding: `F=fft2(f,PQ(1),PQ(2)).`

Basic steps (3-6) in DFT filtering

3. Generate a filter function H of size $PQ1$ x $PQ2$. If the filter function is centered, let `H=fftshift(H)` before using the filter.
4. Multiply the transform by the filter: `G=H.*F;`
5. Obtain the real part of the inverse FFT of G: `g=real(ifft2(G));`
6. Crop the top left rectangle of the original size: `g=g(1:size(f,1),1:size(f,2));`

Filtering procedure summarized

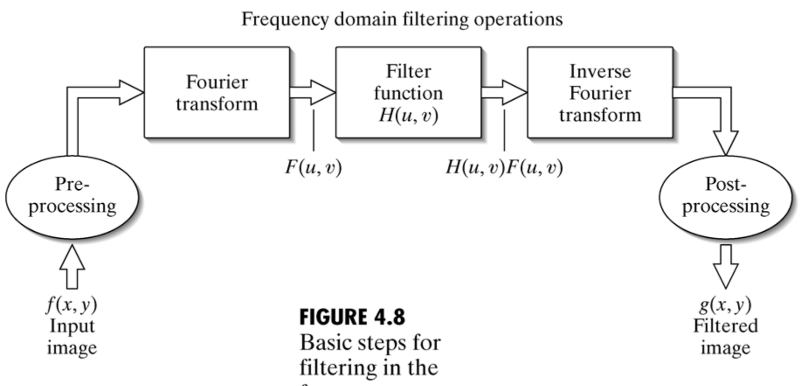


FIGURE 4.8 Basic steps for filtering in the frequency domain.

Converting spatial filters into equivalent frequency domain filters

- Choice between filtering in spatial or frequency domain may depend on the computational efficency
- Filter in the frequency domain:

  `H=freqz2(h,R,C)`,

  where h is a 2D spatial filter,

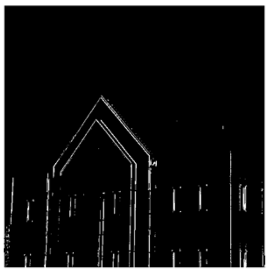
  R is the number of rows and

  C is the number of columns that we wish filter H to have

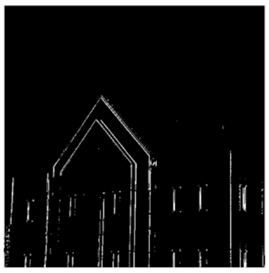
Spatial filtering (Sobel) versus equivalent frequency domain filtering:

identical results

initial image



spatial filtering



frequency domain filtering

(thresholded results)

Creating meshgrid arrays for filters in the frequency domain

```
function [U V]=dftuv(M,N)
% DFTUV Computes meshgrid frequency matrices. U and V are both M-by-N
% Set up range of variables
u=0:(M-1); v=0:(N-1);
% Compute the indices for use in meshgrid
idx=find(u>M/2); u(idx)=u(idx)-M;
idy=find(v>N/2); v(idy)=v(idy)-N;
% Compute the meshgrid arrays
[U V]=meshgrid(v,u);
```

Lowpass frequency domain filters – (1)

* Ideal lowpass filter (ILPF) has the transfer function

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)> D_0\end{cases}$$

where D_0 – specified nonnegative number,

$D(u,v)$ – distance from point (u,v) to the center of the filter

- Ideal filter "cuts off" (multiplies by 0) all components of F outside the circle and leaves unchanged (multiplies by 1) all components on, or inside, the circle

Lowpass frequency domain filters – (2)

* Butterworth lowpass filter (BLPF) of order n, with a cutoff frequency at a distance D_0 from the origin, has the transfer function

$$H(u,v)=\frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

– BLPF transfer function does not have a sharp discontinuity at D_0

Lowpass frequency domain filters – (3)

* Transfer function of a Gaussian lowpass filter (GLPF)

$$H(u,v)=e^{-D^2(u,v)/2\sigma^2}$$

where σ – standard deviation

– By letting $\sigma = D_0$, we obtain the expression for GLPF in terms of the cutoff parameter:

$$H(u,v)=e^{-D^2(u,v)/2D_0^2}$$

Gaussian lowpass filter: example – (1)

```
PQ=2*size(f);
% Create meshgrid for the transfer function – you need dftuv.m
[U V]=dftuv(PQ(1),PQ(2));
D=sqrt(U.^2+V.^2);
% define the parameter for cut-off frequency D0
D0=0.05*PQ(2);  % 5% of the padded image width
% Perform FFT of the original image and obtain the filter transfer function
F=fft2(f,PQ(1),PQ(2));
H=exp(-(D.^2)/(2*(D0^2)));
% Obtain and crop the resulting image
g=real(ifft2(H.*F));
g=g(1:size(f,1),1:size(f,2));
```

Gaussian lowpass filter: example – (2)

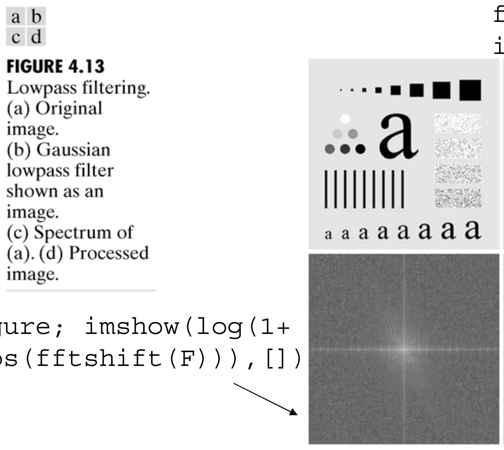


FIGURE 4.13 Lowpass filtering. (a) Original image. (b) Gaussian lowpass filter shown as an image. (c) Spectrum of (a). (d) Processed image.

```
figure;
imshow(fftshift(H),[])
```

```
figure; imshow(log(1+
+abs(fftshift(F))),[])
```

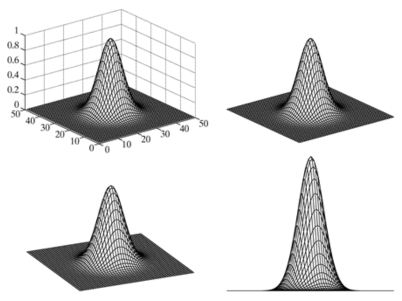
```
figure;
imshow(g,[])
```

Blurred version of the original

3D plots – (1)

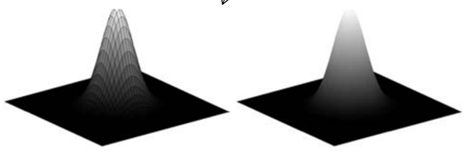
- Draw a plot: **`mesh`**, **`surf`**
- Set or switch off axis: **`axis`**
- Switch on or off the grid: **`grid`**
- Change the viewing point:
  - `view`
  - Click on the “Rotate 3D” button in the figure window’s toolbar

3D plots – (2)



a b
c d

FIGURE 4.15 (a) A plot obtained using function `mesh`. (b) Axes and grid removed. (c) A different perspective view obtained using function `view`. (d) Another view obtained using the same function.



a b

FIGURE 4.16 (a) Plot obtained using function `surf`. (b) Result of using the command `shading interp`.

Sharpening frequency domain filters: highpass filtering

- Highpass filtering sharpens the image by attenuating the low frequencies and leaving the high frequencies of the Fourier transform relatively unchanged
- Transfer function of a highpass filter:

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

where $H_{lp}(u,v)$ – transfer function of the corresponding lowpass filter

Highpas filters: examples

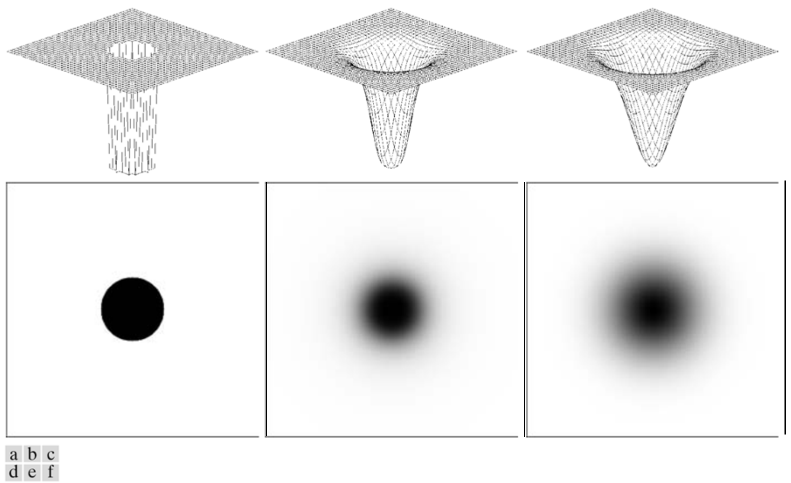
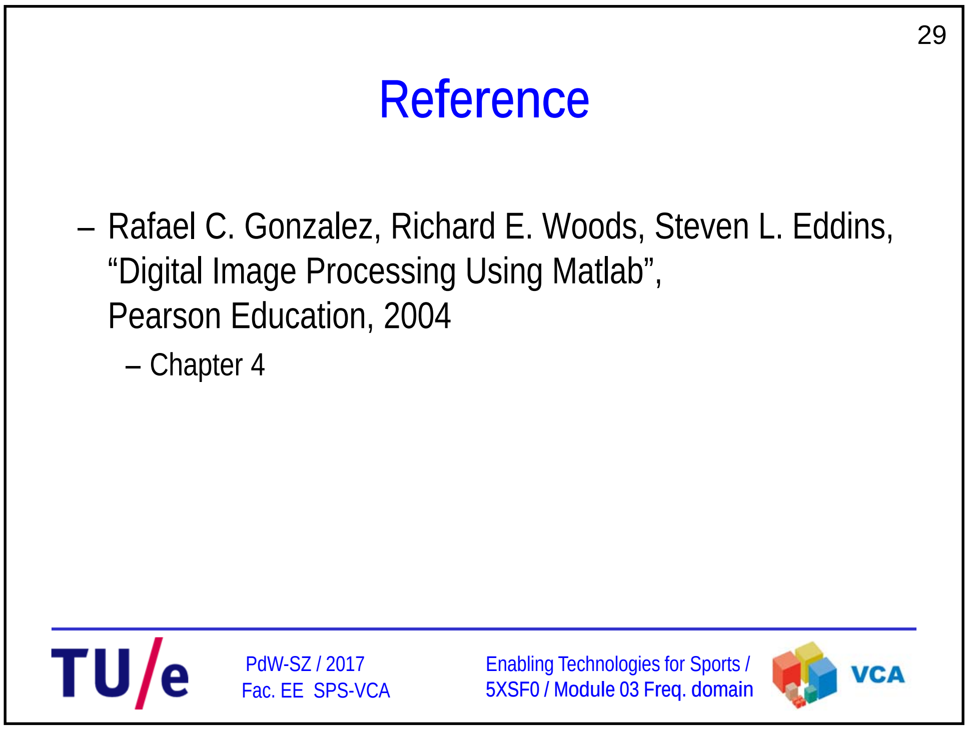


FIGURE 4.17 Top row: Perspective plots of ideal, Butterworth, and Gaussian highpass filters. Bottom row: Corresponding images.



Reference

- Rafael C. Gonzalez, Richard E. Woods, Steven L. Eddins, “Digital Image Processing Using Matlab”, Pearson Education, 2004
  - Chapter 4