Techniques for Video Compression and Analysis (5LSE0), Module 01 - B

Measure of Information

Coding of Discrete Sources

Peter H.N. de With
(p.h.n.de.with@tue.nl)

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Techniques for Video Compression and Analysis (5LSE0), Module 01 - B

Mod 01 - B, Part 1
Probability and Information

Questions to be Answered

Three questions play a central role:

1. Why can signals be compressed?
2. How much can signals be compressed?
3. Which signal processing / information theory algorithms are most efficient in reaching the maximum compression?

(For lossless and *lossy* compression)

Why can Signals be Compressed? – (1)

Because signal amplitudes are mutually dependent

Question 1:

What is the *best possible* exploitation of the correlation (dependencies) in natural signals?

(Rate-Distortion Theory)

Question 2:

How do we *implement* a system that exploits the correlation in natural signals?

(Compression algorithms: - DPCM
- Subband/wavelet
- Transform/DCT
- Motion compensation)

Why can Signals be Compressed? – (2)

Because signal amplitudes are statistically redundant

Question 1:

What is the *shortest* average codeword length that one can achieve for a given signal (or "source")?

(Shannon Information Theory)

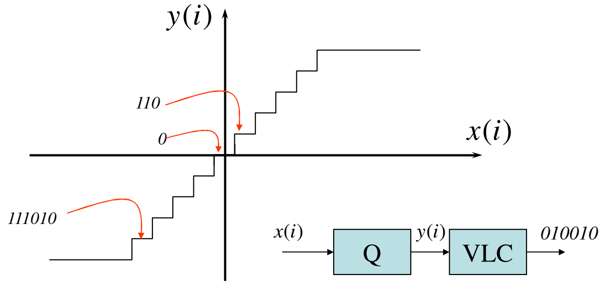
Question 2:

How did you obtain those codewords?

(Construction Recipes)

Relevance of Coding Subject / all corner stones can be re-used…

– Theory is more generally applicable
– As a start, keep in mind that we are discussing how to convert the output of a quantizer to codewords



Example of information: Lottery

What does "Statistical Redundancy" mean?

* **Case 1**: Two people put €10 each on the table. A fair coin is flipped, the winner takes all
* **Case 2A**: 1024 people put €10 each on the table. A number between 1 and 1024 is drawn randomly, the winner takes all.
* **Case 2B**: 1024 people put €10 each on the table, a fair coin is flipped
  * If head, I take all money If tail, I loose. Then a number between 1 and 1023 is drawn randomly, the winner takes all
* **Which case is most surprising ~ has most information?**
* **Can we quantify the term information?**

Information and Probability

* **We want to quantify the concept of *Information***

  Foundation in 1948 by Claude Shannon, ("Mathematical Theory",BSTJ)
* **The *less likely* an event or *symbol* s_i is, the more *uncertainty* exists, and the *more information* one obtains if this event/symbol occurs**

$$I(s_i) = -\log_2[P_S(s_i)] \qquad \text{(Self-information) [bits]}$$

* Case 1: I(win) = 1 bit
* Case 2A: I(win) = 10 bits
* Case 2B: I(I win) = 1 bit / (you win) ≈ 11 bits

Shannon's Measure of Information / Entropy

* **We are interested in the *average* amount of information that one observes per symbol (*average amount of information that a quantizer produces*)**

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bit/symbol)}$$

* **Commonly known as**
  - "p log p" information measure
  - *Entropy* of a source (quantizer) ~ *chaos in physics*

Lottery Example / Sense of information – (2)

* **The *more equally probable* events or symbols s_i are, the *more uncertainty* exist, and *more information* is obtained**

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bit)}$$

* **Case 1:** $H(S)$ = 1 bit
* **Case 2A:** $H(S)$ = 10 bits
* **Case 2B:** $H(S)$ ≈ 6 bits

Another Example / 8-Message source

s_i	$P_S(s_i)$	$I(s_i)$
0	0.125	3 bits
1	0	
2	0.5	1 bits
3	0	
4	0.125	3 bits
5	0.125	3 bits
6	0	
7	0.125	3 bits

$H(S)$ = 2 bits of information per amplitude

Key properties of entropy *H*(*S*)

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \qquad \text{(bits/symb.)}$$

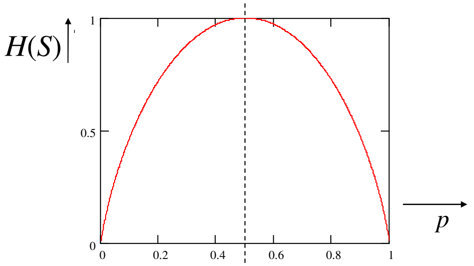
The definition of Information holds for zero-memory (memoryless) discrete sources

* ***H*(*S*) is positive**
* ***H*(*S*) is continuous in symbol probabilities**
* ***H*(*S*) is symmetric**
* ***H*(*S*) is maximum, if all symbol probabilities are equal**

$$0 \le H(S) \le \log_2(N)$$

Entropy example for a Binary Source

* Two symbols s_0 and s_1
* $P[s_0] = p \qquad P[s_1] = 1\text{-}p$
* $H(S) = -p \log_2 p - (1\text{-}p)\log_2(1\text{-}p)$



Theory and practice… memoryless sources

* The expressions discrete / analogue source entropy

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \quad \text{(bit)}$$

$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx$$

hold for memoryless sources.

- Image/video data we wish to compress are usually not memoryless!
- However, we will see "Transform block" will try to do just that

Back to Discrete Sources

* Important application of Shannon's entropy measure is in finding efficient (~ short average length) code words
* The entropy $H(S)$ tells us what the ***minimal average code word length*** is of any
  - instantaneously decodable
  - uniquely decodable
  - nonsingular
  - binary block

  code that can be designed for the source ***S***

  (*without telling how to find that code*)

Techniques for Video Coding and Analysis (5LSE0), Module 01 - B

Mod 01 – B, Part 2

Coding: Definitions & Examples

Back to terms

Example Uniquely Decodable

Not uniquely decodable

s_1	: 0
s_2	: 11
s_3	: 00
s_4	: 01
$s_1 s_3$	: 000
$s_3 s_1$	: 000

Some concatenations lead to a singular code!

Uniquely decodable

s_1	: 0
s_2	: 10
s_3	: 110
s_4	: 111

Any combination of symbols can be uniquely decoded (any concatenation leads to a nonsingular code)

Example Uniquely & Instantaneously Decodable

Both codes are uniquely decodable but other propeorties count as well !

Non-instantaneous

s_1	: 0
s_2	: 01
s_3	: 011
s_4	: 0111

$s_1 s_3 s_4$: 00110111

Need to observe next "0" to know that previous bit ended a code word

Instantaneous

s_1	: 0
s_2	: 10
s_3	: 110
s_4	: 1110

$s_1 s_3 s_4$: 01101110

By observing this "0" we immediately know a code word was found

Shannon: Noiseless Source Coding Theorem

- **For a zero-memory discrete source with entropy**

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \quad \text{(bit)}$$

an (instantaneously and uniquely decodable, nonsingular block) binary code exists for which the average code word length L is

$$H(S) \le L \le H(S) + 1$$

Entropy has even better bound for L … by creating groups of symbols

- **If we group M independent symbols together, then**

$$H(S) \le L \le H(S) + \frac{1}{M}$$

- **In other words, at sufficient expense ($M \to \infty$), a code exists of which the average code word length is arbitrarily close to the entropy of the source.**

Example – 8-Message source

s_i	$P_S(s_i)$
0	*0.005*
1	*0.02*
2	*0.14*
3	*0.20*
4	*0.51*
5	*0.08*
6	*0.04*
7	*0.005*

- **Simple binary coding requires 3 bits/symbol**
- ***H*(*S*) = 2.024 bits/symbol (creates clear reduction!)**

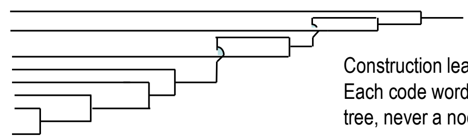
How to find that Code…?

- **A number of construction recipes are know, usually named after their "inventor"**
  - Shannon-Fano code
  - Gilbert-Moore code
  - Arithmetic code
  - **Huffman code**
    - This one is used very often in compression
    - Often in combination with run-length code

Huffman Binary Code Construction – (1)

1. **Rank the symbols with decreasing probability**
2. **Join the two least probable symbols and add their probabilities to form a new "joined symbol"**
3. **Re-arrange new set of probabilities in decreasing order**
4. **Repeat Step 2 and 3 until two probabilities remain**
5. **Assign a bit "0" to one of the probabilities, and a bit "1" to the other**
6. **Go backwards and add one bit at each place where two symbols were joined**
7. **Create code words following the path to that symbol from right to left**

Huffman Binary Code Construction – (2)



Construction leads to a tree!
Each code word is a leaf of the tree, never a node!

s_i	$P_S[s_i]$							
4	0.51						0.51 (0)	0
3	0.20					0.29 (0)	0.49 (1)	11
2	0.14				0.15 (0)	0.20 (1)		101
5	0.08			0.08 (0)	0.14 (1)			1000
6	0.04		0.04 (0)	0.07 (1)				10010
1	0.02	0.02 (0)	0.03 (1)					100110
7	0.005 (0)	0.01 (1)						1001110
0	0.005 (1)							1001111

Huffman Binary Code Construction – (3)

s_i	$P_S(s_i)$	Codeword
0	0.005	1001111
1	0.02	100110
2	0.14	101
3	0.20	11
4	0.51	0
5	0.08	1000
6	0.04	10010
7	0.005	1001110

* **Simple binary coding requires 3 bits/symbol**
* $H(S)$ **= 2.024 bits/symbol**
* L_{av} **= 2.204 bits/symbol** (we approach closely the bound: ... Entropy)

Computing the average code word length

The average code word length is the real practical number !

Average codeword length L**:**

$$L = \sum_{i=1}^{N} P_S(s_i) l_i$$

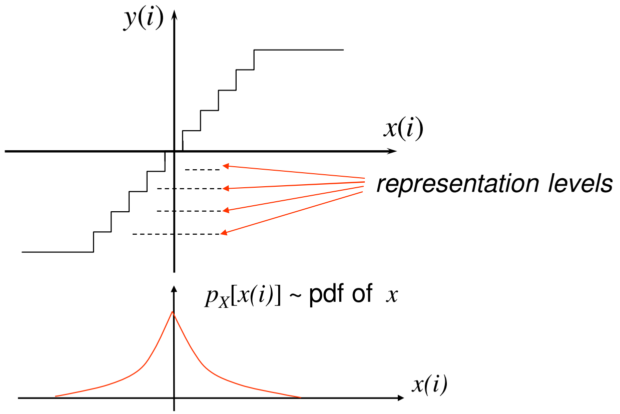
"#Symbols"

Probability of symbol

Length of codeword

Signal value	$P_S(s_i)$	Codeword	l_i
0	0.125	100	3
1	0		-
2	0.5	0	1
3	0		-
4	0.125	101	3
5	0.125	110	3
6	0		-
7	0.125	111	3

Quantization / Example – (1)



Quantization / Example – (2)

	Repres. Level	Probability	Codewords (positive)	Codewords (negative)
1	±9.247	0.00007	00010101101010	00010101101011
2	±6.875	0.00031	000101011011	0001010110100
3	±5.519	0.00077	0001010111	00010101100
4	±4/562	0.00152	0001010000	0001010001
5	±3.823	0.00269	000101001	000101010
6	±3.217	0.00444	00011010	00011011
7	±2.703	0.00699	1011010	1011011
8	±2.253	0.01068	0001011	0001100
9	±1.855	0.01590	000111	101100
10	±1.497	0.02318	10111	000100
11	±1.175	0.03316	01000	01001
12	±0.884	0.04658	1100	1101
13	±0.622	0.06441	0101	1010
14	±0.388	0.08797	111	0000
15	±0.180	0.11987	011	100
16	0.000	0.16292	001	

* **Simple binary coding requires 5 bits/repr.level**
* $H(S)$ **= 3.876 bits/repr.level**
* L_{av} **= 3.912 bits/repr.level**

What if statistical dependencies exist?

- **So far, source symbols were assumed independent**
  - We considered only **discrete *memoryless*** sources
  - Quantizer representation levels were considered one-by-one.
- **What if the source symbols are dependent?**
  - Model that dependency and design codes (~document compression, Markov chains, "context coders")
  - Example solution for coding: use ***runlength*** coding, or in general case: do something else….. (see next slide)

Discussion & Application runlength coding

- **Later, we see that efficient "transforms" used in compression produce**
  - A lot of "zero" values & some (significant) non-zero values

  Solution: grouping of symbols in the coding
  - Otherwise: transform the such that essential information is concentrated (like PCA) or signal is better modeled (wavelet)
- **Example coding: Typical symbol sequences to be coded:**
  "5 1 0 0 0 0 0 0 0 3 0 0 6 0 0 0 0 1 0 0 0 0 …" (essential info?)
  - Will be done by {zero-run, non-zero symbol} pairs
  - Here: "{0,5}, {0,1}, {7,3}, {2,6}, {4,1}, …"
  - The pairs are assigned a Huffman code

Why can Signals be Compressed?

Because ~~signal amplitudes are~~ statistically redundant
quantizer representation levels

Question 1:
What is the *shortest* average codeword length that one can achieve for a given signal (or "source")?

$$H(S) = -\sum_{i=1}^{N} P_S(s_i)\log_2[P_S(s_i)] \quad \text{(bit)}$$

$$H(X) = -\int_{-\infty}^{\infty} p_X(x)\log_2[p_X(x)]dx$$

Question 2:
How did you obtain those codewords?

Huffman Coding
Run-length coding